

Graph homomorphisms on rectangular matrices over division rings II*

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Abstract

Let $\mathbb{D}^{m \times n}$ be the set of $m \times n$ matrices over a division ring \mathbb{D} . Two matrices $A, B \in \mathbb{D}^{m \times n}$ are adjacent if $\text{rank}(A - B) = 1$. By the adjacency, $\mathbb{D}^{m \times n}$ is a connected graph. Suppose \mathbb{D}, \mathbb{D}' are division rings and $m, n, m', n' \geq 2$ are integers. We determine additive graph homomorphisms from $\mathbb{D}^{m \times n}$ to $\mathbb{D}'^{m' \times n'}$. When $|\mathbb{D}| \geq 4$, we characterize the graph homomorphism $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ if $\varphi(0) = 0$ and there exists $A_0 \in \mathbb{D}^{n \times n}$ such that $\text{rank}(\varphi(A_0)) = n$. We also discuss properties and ranges on degenerate graph homomorphisms. If $f : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ (where $\min\{m, n\} = 2$) is a degenerate graph homomorphism, we prove that the image of f is contained in a union of two maximal adjacent sets of different types. For the case of finite fields, we obtain two better results on degenerate graph homomorphisms.

Keywords: graph homomorphism, matrix, division ring, additive graph homomorphism, degenerate graph homomorphism, geometry of matrices

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1 Introduction

The study of the *geometry of matrices* was initiated by Hua in the mid 1940s [8, 26]. The fundamental problem in the geometry of matrices is to characterize the transformation group of matrices by as few geometric invariants as possible. In the view of equivalence, the basic problem of the geometry of matrices is also to study graph isomorphisms on matrices. In 1951, Hua [8] proved the fundamental theorem of the geometry of rectangular matrices over division rings. Hua's theorem also characterized the graph isomorphism on rectangular matrices, and his work was continued by many scholars (cf. [3], [9]-[19], [21], [23]-[28]).

In the algebraic graph theory, the research of graph homomorphisms is an important subject [6, 7]. Thus, it is of significance to determine graph homomorphisms on matrices. Recently, literatures [24, 17, 23, 13, 14] discussed the graph homomorphisms on matrices over a division ring, and these work is interesting for the geometry of matrices, the algebraic graph theory and the preserver problems. This paper is continuation of literature [14], and our goal is further to characterize the graph homomorphism on rectangular matrices. All mathematical symbols and definitions that are not explained, see [14].

Throughout this paper, we assume that \mathbb{D}, \mathbb{D}' are division rings, $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, and m, n, m', n' are positive integers. Denote by \mathbb{F}_q the finite field with q elements where q is a power of a prime. Let $|X|$ be the cardinality of a set X . Let \mathbb{D}^n [resp. ${}^n\mathbb{D}$] denote the left [resp. right] vector space over \mathbb{D} whose elements are n -dimensional row [resp. column] vectors over \mathbb{D} . On the basic properties of matrices over a division ring, one may see literatures [20, 26].

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Let $\mathbb{D}^{m \times n}$ and $\mathbb{D}_r^{m \times n}$ denote the sets of $m \times n$ matrices and $m \times n$ matrices of rank r over \mathbb{D} , respectively. A matrix in $\mathbb{D}^{m \times n}$ is also called a point. Denote the set of $n \times n$ invertible matrices over \mathbb{D} by $GL_n(\mathbb{D})$. Let I_r (I for short) be the $r \times r$ identity matrix, $0_{m,n}$ the $m \times n$ zero matrix (0 for short) and $0_n = 0_{n,n}$. Let $E_{ij}^{m \times n}$ (E_{ij} for short) denote the $m \times n$ matrix whose (i, j) -entry is 1 and all other entries are 0's. Denote by ${}^t A$ the transpose matrix of a matrix A . If $\sigma : \mathbb{D} \rightarrow \mathbb{D}'$ is a map and $A = (a_{ij}) \in \mathbb{D}^{m \times n}$, we write $A^\sigma = (a_{ij}^\sigma)$ and ${}^t A^\sigma = {}^t (A^\sigma)$.

Let $\Gamma(\mathbb{D}^{m \times n})$ be the graph whose vertex set is $\mathbb{D}^{m \times n}$ and two vertices $A, B \in \mathbb{D}^{m \times n}$ are adjacent if $\text{rank}(A - B) = 1$. We write $A \sim B$ if $\text{rank}(A - B) = 1$. The $\Gamma(\mathbb{D}^{m \times n})$ is called the graph on $m \times n$ matrices over \mathbb{D} . $\Gamma(\mathbb{D}^{m \times n})$ is a connected distance transitive graph [6]. When $\mathbb{D} = \mathbb{F}_q$, $\Gamma(\mathbb{F}_q^{m \times n})$ is also called a *bilinear forms graph* [2]. For $A, B \in \mathbb{D}^{m \times n}$, $d(A, B) := \text{rank}(A - B)$ is the *distance* between A and B in $\Gamma(\mathbb{D}^{m \times n})$ (cf. [20, 26]). The *diameter* of a subgraph G of $\Gamma(\mathbb{D}^{m \times n})$, denoted by $\text{diam}(G)$, is the maximum distance between two distinct vertices in G .

Let $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ be a map. The φ is called a *graph homomorphism* if $A \sim B$ implies that $\varphi(A) \sim \varphi(B)$. The φ is called a *distance preserving map* if $d(A, B) = d(\varphi(A), \varphi(B))$ for all $A, B \in \mathbb{D}^{m \times n}$. The φ is called a *distance k preserving map* if $d(A, B) = k$ implies that $d(\varphi(A), \varphi(B)) = k$ for some fixed k . In the geometry of matrices, a graph homomorphism [resp. *graph isomorphism*] is also called an *adjacency preserving map* [resp. *adjacency preserving bijection in both directions*]. If $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism, then

$$d(\varphi(A), \varphi(B)) \leq d(A, B), \text{ for all } A, B \in \mathbb{D}^{m \times n}. \quad (1)$$

A nonempty subset \mathcal{S} of $\mathbb{D}^{m \times n}$ is called an *adjacent set* if any two distinct vertices (matrices) in \mathcal{S} are adjacent. An adjacent set \mathcal{M} in $\mathbb{D}^{m \times n}$ is called a *maximal adjacent set* (*maximal set* for short), if there is no adjacent set in $\mathbb{D}^{m \times n}$ which properly contains \mathcal{M} as a subset. In graph theory, a maximal set is also called a *maximal clique* [2, 6]. In $\mathbb{D}^{m \times n}$, every adjacent set can be extended to a maximal set, and there are only two type of maximal sets. For convenience, we think that a maximal set and its vertex set are equal.

Suppose that \mathbb{D} and \mathbb{D}' are division rings and $m, n, m', n' \geq 2$ are integers. Write $E_{ij} = E_{ij}^{m \times n}$ and $E'_{ij} = E'_{ij}^{m' \times n'}$. For $1 \leq i \leq m$ and $1 \leq j \leq n$ (or $1 \leq i \leq m'$ and $1 \leq j \leq n'$), we let

$$\mathcal{M}_i = \left\{ \sum_{j=1}^n x_j E_{ij} : x_j \in \mathbb{D} \right\}, \quad \mathcal{N}_j = \left\{ \sum_{i=1}^m y_i E_{ij} : y_i \in \mathbb{D} \right\} \subset \mathbb{D}^{m \times n}, \quad (2)$$

$$\mathcal{M}'_i = \left\{ \sum_{j=1}^{n'} x_j E'_{ij} : x_j \in \mathbb{D}' \right\}, \quad \mathcal{N}'_j = \left\{ \sum_{i=1}^{m'} y_i E'_{ij} : y_i \in \mathbb{D}' \right\} \subset \mathbb{D}'^{m' \times n'}. \quad (3)$$

Lemma 1.1 (cf. [26, Proposition 3.8]) *Every maximal set \mathcal{M} of $\mathbb{D}^{m \times n}$ is one of the following forms.*

Type one. $\mathcal{M} = P\mathcal{M}_1Q + A = P\mathcal{M}_1 + A$, where $P \in GL_m(\mathbb{D})$, $Q \in GL_n(\mathbb{D})$ and $A \in \mathbb{D}^{m \times n}$.

Type two. $\mathcal{M} = P\mathcal{N}_1Q + A = \mathcal{N}_1Q + A$, where $P \in GL_m(\mathbb{D})$, $Q \in GL_n(\mathbb{D})$ and $A \in \mathbb{D}^{m \times n}$.

For $1 \leq k \leq \min\{m, n\}$ and $A \in \mathbb{D}^{m \times n}$, we let

$$\mathbb{D}_{\leq k}^{m \times n} = \{X \in \mathbb{D}^{m \times n} : \text{rank}(X) \leq k\},$$

$$\mathbb{B}_A = \mathbb{D}_{\leq 1}^{m \times n} + A,$$

$$\mathbb{N}_A = \mathbb{D}_1^{m \times n} + A \subset \mathbb{B}_A.$$

The \mathbb{B}_A is called the *unit ball* with a central point A , and the \mathbb{N}_A is called the *neighbourhood* of A . If $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}^{m' \times n'}$ is a graph homomorphism, then $\varphi(\mathbb{N}_A) \subseteq \mathbb{N}_{\varphi(A)}$ and $\varphi(\mathbb{B}_A) \subseteq \mathbb{B}_{\varphi(A)}$ for every $A \in \mathbb{D}^{m \times n}$.

Let $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}^{m' \times n'}$ be a graph homomorphism. The homomorphism φ is called *degenerate*, if there exists a matrix $A \in \mathbb{D}_{\leq 1}^{m \times n}$ and there are two maximal sets \mathcal{M} and \mathcal{N} of different types in $\mathbb{D}^{m' \times n'}$, such that $\varphi(\mathbb{B}_A) \subseteq \mathcal{M} \cup \mathcal{N}$ with $\varphi(A) \in \mathcal{M} \cap \mathcal{N}$. The homomorphism φ is called *non-degenerate* if it is not degenerate. The homomorphism φ is called a (vertex) *colouring* if $\varphi(\mathbb{D}^{m \times n})$ is an adjacent set in $\mathbb{D}^{m' \times n'}$.

Every (vertex) colouring is a degenerate graph homomorphism. It is easy to see that φ is a (vertex) colouring if and only if $\text{diam}(\varphi(\mathbb{D}^{m \times n})) = 1$. The word “colouring” is derived from graph theory [6]. Any distance preserving map is a non-degenerate graph homomorphism but not vice versa.

This paper is organized as follows. In Section 2, we introduce some Lemmas on maximal sets. In Section 3, We determine additive graph homomorphisms from $\mathbb{D}^{m \times n}$ to $\mathbb{D}^{m' \times n'}$. In Section 4, we will characterize the graph homomorphism φ from $\mathbb{D}^{m \times n}$ to $\mathbb{D}^{m' \times n'}$ (where $|\mathbb{D}| \geq 4$) if there exists $A_0 \in \mathbb{D}^{m \times n}$ such that $\text{rank}(\varphi(A_0)) = n$, this result extends [24, Theorem 4.1] to the cases of two division rings and $n = 2$. In Section 5, we discuss the degenerate graph homomorphisms. Let $f : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}^{m' \times n'}$ (where $|\mathbb{D}| \geq 4$) be a degenerate graph homomorphism. We prove that $f(\mathbb{D}_{\leq 1}^{m \times n})$ and $f(\mathbb{B}_{A_0})$ are two adjacent sets, if there exists $A_0 \in \mathbb{D}^{m \times n}$ such that $\text{rank}(f(A_0)) = \min\{m, n\}$. Moreover, when $\min\{m, n\} = 2$, the image of f is contained in a union of two maximal sets of different types. For the case of finite fields, we obtain two better results on the degenerate graph homomorphisms.

2 Lemmas on maximal sets

In this section, we will introduce maximal sets on $\mathbb{D}^{m \times n}$ and their affine geometries.

Lemma 2.1 (cf. [9, Lemma 3.2]) *Let m, n, r, s be integers with $1 \leq r, s < \min\{m, n\}$. Assume that $\alpha = \{i_1, \dots, i_r\}$, $\beta = \{j_1, \dots, j_s\}$, where $1 \leq i_1 < \dots < i_r \leq m$ and $1 \leq j_1 < \dots < j_s \leq n$. Let $A = (a_{ij}) \in \mathbb{D}^{m \times n}$, $B_i = \sum_{r=1}^r \sum_{k=1}^s b_{i,j_k}^{(i)} E_{i,j_k} \in \mathbb{D}^{m \times n}$ (where $b_{i,j_k}^{(i)} \in \mathbb{D}$), $i = 1, 2$, and $B_1 \neq B_2$. If $A \sim B_i$, $i = 1, 2$, then either $a_{ij} = 0$ for all $i \notin \alpha$, or $a_{ij} = 0$ for all $j \notin \beta$.*

Using Lemmas 2.1 and 1.1, we can prove the following results.

Corollary 2.2 (cf. [26, Corollary 3.10]) *Let A and B be two adjacent points in $\mathbb{D}^{m \times n}$. Then there are exactly two maximal sets \mathcal{M} and \mathcal{M}' containing A and B . Moreover, \mathcal{M} and \mathcal{M}' are of different types.*

Corollary 2.3 (cf. [9, 12, 26]) *If \mathcal{M} and \mathcal{N} are two distinct maximal sets of the same type [resp. different types] in $\mathbb{D}^{m \times n}$ with $\mathcal{M} \cap \mathcal{N} \neq \emptyset$, then $|\mathcal{M} \cap \mathcal{N}| = 1$ [resp. $|\mathcal{M} \cap \mathcal{N}| \geq 2$].*

Lemma 2.4 (cf. [14, Lemma 3.4]) *Suppose that \mathcal{M}, \mathcal{N} are two distinct maximal sets in $\mathbb{D}^{m \times n}$ with $\mathcal{M} \cap \mathcal{N} \neq \emptyset$. Then:*

(i) *if \mathcal{M}, \mathcal{N} are of different types, then for any $A \in \mathcal{M} \cap \mathcal{N}$, there are $P \in GL_m(\mathbb{D})$ and $Q \in GL_n(\mathbb{D})$ such that $\mathcal{M} = PM_1Q + A = PM_1 + A$ and $\mathcal{N} = PN_1Q + A = N_1Q + A$;*

(ii) if both \mathcal{M} and \mathcal{N} are of type one [resp. type two], then there exists an invertible matrix P [resp. Q] such that $\mathcal{M} = PM_1 + A$ and $\mathcal{N} = PM_2 + A$ [resp. $\mathcal{M} = N_1Q + A$ and $\mathcal{N} = N_2Q + A$], where $\mathcal{M} \cap \mathcal{N} = \{A\}$.

There is an axiomatic definition of the *affine geometry* (cf. [1, 5]). Let V be an r -dimensional left vector subspace of \mathbb{D}^n and $a \in \mathbb{D}^n$. Then $V + a$ is called an r -dimensional *left affine flat* (flat for short) over \mathbb{D} . When $r \geq 2$, the set of all flats in $V + a$ is called the *left affine geometry* on $V + a$, which is denoted by $AG(V + a)$. The *dimension* of $AG(V + a)$ is r , denoted by $\dim(AG(V + a)) = r$. The flats of dimensions 0, 1, 2 are called *points*, *lines*, *planes* in $AG(V + a)$. Similarly, we have the *right affine geometry* on a right affine flat over \mathbb{D} .

Let $\mathcal{M} = PM_1 + A$ be a maximal set of type one in $\mathbb{D}^{m \times n}$, where $P \in GL_m(\mathbb{D})$ and $A \in \mathbb{D}^{m \times n}$. Then we have a left affine geometry $AG(PM_1 + A)$ such that $AG(PM_1 + A)$ and $AG(\mathbb{D}^n)$ are affine isomorphic. Similarly, we have a right affine geometry $AG(N_1Q + A)$ such that $AG(N_1Q + A)$ and $AG({}^m\mathbb{D})$ are affine isomorphic, where $Q \in GL_n(\mathbb{D})$ (cf. [14]). In $AG(PM_1 + A)$, the parametric equation of a line ℓ is

$$\ell = P \begin{pmatrix} \mathbb{D}\alpha + \beta \\ 0 \end{pmatrix} + A, \text{ where } \alpha, \beta \in \mathbb{D}^n \text{ with } \alpha \neq 0.$$

Lemma 2.5 (cf. [26, p.95]) *Let \mathcal{M} be a maximal set in $\mathbb{D}^{m \times n}$. Then ℓ is a line in $AG(\mathcal{M})$ if and only if $\ell = \mathcal{M} \cap \mathcal{N}$, where \mathcal{M} and \mathcal{N} are two maximal sets of different types with $\mathcal{M} \cap \mathcal{N} \neq \emptyset$. Moreover, $|\ell| = |\mathcal{M} \cap \mathcal{N}| = |\mathbb{D}|$.*

Lemma 2.6 (see [14, Lemma 4.6]) *Let $m, n, m', n' \geq 2$ be integers. Suppose that $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a non-degenerate graph homomorphism with $\varphi(0) = 0$. Then*

- (a) *if \mathcal{M} is a maximal set of type one [resp. type two] containing 0 in $\mathbb{D}^{m \times n}$ and $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$, where \mathcal{M}' is a maximal set in $\mathbb{D}'^{m' \times n'}$, then for any $A \in \mathcal{M}$, there exists a maximal set \mathcal{R} of type one [resp. type two] in $\mathbb{D}^{m \times n}$, such that $\varphi(\mathcal{R}) \subseteq \mathcal{R}'$ and $\mathcal{R} \cap \mathcal{M} = \{A\}$, where \mathcal{R}' is a maximal set in $\mathbb{D}'^{m' \times n'}$, \mathcal{R}' and \mathcal{M}' are of the same type with $\mathcal{R}' \neq \mathcal{M}'$;*
- (b) *if \mathcal{M} and \mathcal{N} are two distinct maximal sets of the same type [resp. different types] in $\mathbb{D}^{m \times n}$ such that $0 \in \mathcal{M}$ and $\mathcal{M} \cap \mathcal{N} \neq \emptyset$, then $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ and $\varphi(\mathcal{N}) \subseteq \mathcal{N}'$, where \mathcal{M}' and \mathcal{N}' are two maximal sets of the same type [resp. different types] in $\mathbb{D}'^{m' \times n'}$;*
- (c) *if \mathcal{M} is a maximal set containing 0 in $\mathbb{D}^{m \times n}$ and $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ where \mathcal{M}' is a maximal set in $\mathbb{D}'^{m' \times n'}$, then \mathcal{M}' is the unique maximal set containing $\varphi(\mathcal{M})$, and $\varphi(\mathcal{M})$ is not contained in any line of $AG(\mathcal{M}')$.*

Let S be an adjacent set in $\mathbb{D}^{m \times n}$ with $0 \in S$ and $|S| \geq 2$. Then there exists a maximal set \mathcal{M} containing S . By Lemma 1.1, $\mathcal{M} = PM_1$ or $\mathcal{M} = N_1Q$, where P and Q are invertible matrices. Hence $P^{-1}S \subseteq \mathcal{M}_1$ or $SQ^{-1} \subseteq \mathcal{N}_1$. When $P^{-1}S \subseteq \mathcal{M}_1$ [resp. $SQ^{-1} \subseteq \mathcal{N}_1$], the *dimension* of S , denoted by $\dim(S)$, is the number of matrices in a maximal left [resp. right] linear independent subset of $P^{-1}S$ [resp. SQ^{-1}]. The $\dim(S)$ is uniquely determined by S and $\dim(S) \leq \max\{m, n\}$.

Lemma 2.7 (see [14, Lemma 4.13]) *Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$. Suppose $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a non-degenerate graph homomorphism with $\varphi(0) = 0$. Let \mathcal{M} be a maximal set containing 0 in $\mathbb{D}^{m \times n}$. If $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ where \mathcal{M}' is a maximal set in $\mathbb{D}'^{m' \times n'}$, then the restriction map $\varphi|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}'$*

is an injective weighted semi-affine map. Moreover, if \mathcal{S} is an adjacent set in $\mathbb{D}^{m \times n}$ with $0 \in \mathcal{S}$ and $|\mathcal{S}| \geq 2$, then

$$\dim(\varphi(\mathcal{S})) \leq \dim(\mathcal{S}).$$

Lemma 2.8 *Let \mathcal{M} be a maximal set in $\mathbb{D}^{m \times n}$ ($m, n \geq 2$) and $A \in \mathbb{D}^{m \times n}$. Then $A \in \mathcal{M}$ if and only if A is adjacent with three noncollinear points in $AG(\mathcal{M})$.*

Proof. If $A \in \mathcal{M}$, it is clear that A is adjacent with three noncollinear points in $AG(\mathcal{M})$. Now, suppose that A is adjacent with three noncollinear points B_1, B_2, B_3 in $AG(\mathcal{M})$. Then, there are $P \in GL_m(\mathbb{D})$ and $Q \in GL_n(\mathbb{D})$ such that $B_2 - B_1 = PE_{11}Q$. After by the map $X \mapsto P^{-1}(X - B_1)Q^{-1}$, we may assume with no loss of generality that $B_1 = 0$ and $B_2 = E_{11}$. By Corollary 2.2, we have either $\mathcal{M} = \mathcal{M}_1$ or $\mathcal{M} = \mathcal{N}_1$. Without loss of generality we assume that $\mathcal{M} = \mathcal{M}_1$. Clearly, $\ell = \{xE_{11} : x \in \mathbb{D}\}$ is a line in $AG(\mathcal{M}_1)$ containing 0 and E_{11} . Since $0, E_{11}, B_3$ are noncollinear points in $AG(\mathcal{M}_1)$, B_3 is of the form $B_3 = \begin{pmatrix} b & \beta \\ 0 & 0 \end{pmatrix}$ where $0 \neq \beta \in \mathbb{D}^{n-1}$. Write $A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where $a_{11} \in \mathbb{D}$. By $A \sim 0$, $A \sim E_{11}$ and $A \sim \begin{pmatrix} b & \beta \\ 0 & 0 \end{pmatrix}$, we get

$$\text{rank} \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \text{rank} \begin{pmatrix} a_{11} - 1 & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \text{rank} \begin{pmatrix} a_{11} - b & A_{12} - \beta \\ A_{21} & A_{22} \end{pmatrix} = 1.$$

It follows from Lemma 2.1 that $(A_{21}, A_{22}) = 0$, and hence $A \in \mathcal{M}_1$. \square

3 Additive graph homomorphisms on rectangular matrices

Let $\phi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ be a map. The ϕ is called an *additive map* if $\phi(A + B) = \phi(A) + \phi(B)$ for all $A, B \in \mathbb{D}^{m \times n}$. The ϕ is called an *additive graph homomorphism* if ϕ is a graph homomorphism and an additive map. If ϕ is an additive graph homomorphism, then $\phi(0) = 0$ and $\phi(A - B) = \phi(A) - \phi(B)$ for all $A, B \in \mathbb{D}^{m \times n}$.

The additive preserver problems on matrices is an active research area in linear algebra and the theory of matrices [29]. In the additive preserver problems, an additive graph homomorphism is also called an *additive rank-1 preserving map*. When $\mathbb{D} = \mathbb{D}'$ is a field, additive rank-1 preserving maps from $\mathbb{D}^{m \times n}$ to $\mathbb{D}^{m' \times n'}$ is determined by [29]. In this section, we discuss the case of division rings, and our main result is the following theorem.

Theorem 3.1 *Suppose \mathbb{D}, \mathbb{D}' are division rings and $m, n, m', n' \geq 2$ are integers. Let $\phi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ be an additive graph homomorphism. Then either there exist matrices $P \in \mathbb{D}'^{m' \times m}$ and $Q \in \mathbb{D}^{n \times n'}$ with their ranks ≥ 2 , and a nonzero ring homomorphism $\tau : \mathbb{D} \rightarrow \mathbb{D}'$, such that*

$$\phi(X) = PX^\tau Q, \quad X \in \mathbb{D}^{m \times n}; \quad (4)$$

or there exist matrices $P \in \mathbb{D}'^{m' \times n}$ and $Q \in \mathbb{D}^{m \times n'}$ with their ranks ≥ 2 , and a nonzero ring anti-homomorphism $\sigma : \mathbb{D} \rightarrow \mathbb{D}'$, such that

$$\phi(X) = P^t X^\sigma Q, \quad X \in \mathbb{D}^{m \times n}; \quad (5)$$

or ϕ is a (vertex) colouring.

In order to prove Theorem 3.1, we need the following results.

Theorem 3.2 Suppose \mathbb{D}, \mathbb{D}' are division rings and $m, n, m', n' \geq 2$ are integers. Let $\phi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ be an additive graph homomorphism. If ϕ is degenerate, then ϕ is a (vertex) colouring.

Proof. Assume that ϕ is degenerate. Since ϕ is additive, there are two maximal sets \mathcal{M}, \mathcal{N} of different types in $\mathbb{D}'^{m' \times n'}$, such that $\phi\left(\mathbb{D}_{\leq 1}^{m \times n}\right) \subseteq \mathcal{M} \cup \mathcal{N}$ with $0 \in \mathcal{M} \cap \mathcal{N}$. We show that $\phi\left(\mathbb{D}_{\leq 1}^{m \times n}\right)$ is an adjacent set by contradiction as follows.

Suppose $\phi\left(\mathbb{D}_{\leq 1}^{m \times n}\right)$ is not an adjacent set. Then there are matrices $A, B \in \mathbb{D}_1^{m \times n}$ such that $\text{rank}(A - B) = \text{rank}(\phi(A) - \phi(B)) = 2$. Let $E_{ij} = E_{ij}^{m \times n}$ and $E'_{ij} = E_{ij}^{m' \times n'}$. There are invertible matrices P_1, Q_1 over \mathbb{D} such that $A = P_1 E_{11} Q_1$ and $B = P_1 E_{22} Q_1$. Also, there are invertible matrices P_2, Q_2 over \mathbb{D}' such that $P_2 \phi(A) Q_2 = E'_{11}$ and $P_2 \phi(B) Q_2 = E'_{22}$. Replacing ϕ by the map $X \mapsto P_2 \phi(P_1 X Q_1) Q_2$, we have

$$\phi(E_{ii}) = E'_{ii}, \quad i = 1, 2. \quad (6)$$

By Corollary 2.2, there are exactly two maximal sets containing E_{ii} and 0 [resp. E'_{ii} and 0], they are \mathcal{M}_i and \mathcal{N}_i [resp. \mathcal{M}'_i and \mathcal{N}'_i] ($i = 1, 2$). Thus, by (6) and $\phi(0) = 0$, we get that $\phi(\mathcal{M}_1) \subseteq \mathcal{M}'_1$ or $\phi(\mathcal{M}_1) \subseteq \mathcal{N}'_1$. Without loss of generality, we assume $\phi(\mathcal{M}_1) \subseteq \mathcal{M}'_1$. From the above discussion we have $\phi(\mathcal{N}_2) \subseteq \mathcal{N}'_2$ or $\phi(\mathcal{N}_2) \subseteq \mathcal{M}'_2$. Since $\mathcal{M}_1 \cap \mathcal{N}_2 = \mathbb{D} E_{12}$ and $\mathcal{M}'_1 \cap \mathcal{M}'_2 = \{0\}$, we must have $\phi(\mathcal{N}_2) \subseteq \mathcal{N}'_2$. By $\mathcal{M}'_1 \cap \mathcal{N}'_2 = \mathbb{D}' E'_{12}$, we obtain $\phi(\lambda E_{12}) = \lambda^* E'_{12}$ for all $\lambda \in \mathbb{D}$. Since ϕ is additive,

$$\phi(-E_{11} - E_{12} - E_{22}) = -E'_{11} + (-1)^* E'_{12} - E'_{22}.$$

Recall that $\phi\left(\mathbb{D}_{\leq 1}^{m \times n}\right) \subseteq \mathcal{M} \cup \mathcal{N}$. Using Corollary 2.2, it is clear that $\phi\left(\mathbb{D}_{\leq 1}^{m \times n}\right) \subseteq \mathcal{M}'_1 \cup \mathcal{N}'_2$. Since $E_{21} \sim (-E_{11} - E_{12} - E_{22})$, $\phi(E_{21}) \sim (-E'_{11} + (-1)^* E'_{12} - E'_{22})$. On the other hand, by $\phi(E_{21}) \in \mathcal{M}'_1 \cup \mathcal{N}'_2$ and $\phi(E_{21}) \sim E'_{ii}$ ($i = 1, 2$), we must have $\phi(E_{21}) = b E'_{12}$ where $b \in \mathbb{D}^*$. Hence

$$\text{rank}(\phi(E_{21}) - (-E'_{11} + (-1)^* E'_{12} - E'_{22})) = 2,$$

a contradiction.

Thus, $\phi\left(\mathbb{D}_{\leq 1}^{m \times n}\right)$ is an adjacent set. Then there is a fixed maximal set \mathcal{M}' containing 0 in $\mathbb{D}'^{m' \times n'}$, such that $\phi\left(\mathbb{D}_{\leq 1}^{m \times n}\right) \subseteq \mathcal{M}'$. Note that $Y_1, Y_2 \in \mathcal{M}'$ implies that $Y_1 + Y_2 \in \mathcal{M}'$. Since ϕ is additive, it is clear that $\phi(\mathbb{D}^{m \times n}) \subseteq \mathcal{M}'$, and hence ϕ is a (vertex) colouring. \square

Lemma 3.3 Suppose \mathbb{D}, \mathbb{D}' are division rings and $m, n, m', n' \geq 2$ are integers. Let $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ be a non-degenerate graph homomorphism with $\varphi(0) = 0$. Then there exist two invertible matrices T_1 and T_2 over \mathbb{D}' , and two invertible matrices $T_3^{-1} = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL_m(\mathbb{D})$ and $T_4^{-1} = \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix} \in GL_n(\mathbb{D})$, such that either

$$\varphi(T_3^{-1} \mathcal{M}_i T_4^{-1}) \subseteq T_1 \mathcal{M}'_i T_2 \quad \text{with} \quad \varphi(T_3^{-1} \mathcal{N}_i T_4^{-1}) \subseteq T_1 \mathcal{N}'_i T_2 \quad (i = 1, 2),$$

or

$$\varphi(T_3^{-1} \mathcal{M}_i T_4^{-1}) \subseteq T_1 \mathcal{N}'_i T_2 \quad \text{with} \quad \varphi(T_3^{-1} \mathcal{N}_i T_4^{-1}) \subseteq T_1 \mathcal{M}'_i T_2 \quad (i = 1, 2).$$

Proof. Case 1. $\varphi(\mathcal{M}_1)$ is contained in a maximal set of type one. Then there exists an invertible matrix T_0 over \mathbb{D}' such that $\varphi(\mathcal{M}_1) \subseteq T_0 \mathcal{M}'_1$. By Lemma 2.6(a), there exists a type one maximal set \mathcal{M} in $\mathbb{D}^{m \times n}$

containing 0, such that $\mathcal{M} \neq \mathcal{M}_1$ and $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$, where \mathcal{M}' is a type one maximal set in $\mathbb{D}'^{m' \times n'}$ containing 0 and $\mathcal{M}' \neq T_0 \mathcal{M}'_1$. Using Lemma 2.4, there exists $T_1 \in GL_{m'}(\mathbb{D}')$ such that $T_0 \mathcal{M}'_1 = T_1 \mathcal{M}'_1$ and $\mathcal{M}' = T_1 \mathcal{M}'_2$. Thus, $\varphi(\mathcal{M}_1) \subseteq T_1 \mathcal{M}'_1$ and $\varphi(\mathcal{M}) \subseteq T_1 \mathcal{M}'_2$. Since

$$\mathcal{M} = \left\{ \begin{pmatrix} a_1 x \\ a_2 x \\ \vdots \\ a_m x \end{pmatrix} : x \in \mathbb{D}^n \right\} \text{ with } \begin{pmatrix} a_2 \\ \vdots \\ a_m \end{pmatrix} \neq 0,$$

there is $T_3^{-1} = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL_m(\mathbb{D})$ such that $\mathcal{M} = T_3^{-1} \mathcal{M}_2$ and $\mathcal{M}_1 = T_3^{-1} \mathcal{M}_1$. Thus

$$\varphi(T_3^{-1} \mathcal{M}_i) \subseteq T_1 \mathcal{M}'_i, \quad i = 1, 2.$$

Similarly, there exists a type two maximal set \mathcal{N} in $\mathbb{D}^{m \times n}$ containing 0, such that $\mathcal{N} \neq \mathcal{N}_1$, $\varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1 T_2 = T_1 \mathcal{N}'_1 T_2$ and $\varphi(\mathcal{N}) \subseteq \mathcal{N}'_2 T_2 = T_1 \mathcal{N}'_2 T_2$, where $T_2 \in GL_{n'}(\mathbb{D}')$. Moreover, there is $T_4^{-1} = \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix} \in GL_n(\mathbb{D})$ such that $\mathcal{N} = \mathcal{N}_2 T_4^{-1} = T_3^{-1} \mathcal{N}_2 T_4^{-1}$ and $\mathcal{N}_1 = \mathcal{N}_1 T_4^{-1} = T_3^{-1} \mathcal{N}_1 T_4^{-1}$. It follows that $\varphi(T_3^{-1} \mathcal{N}_i T_4^{-1}) \subseteq T_1 \mathcal{N}'_i T_2$, $i = 1, 2$. Clearly, $T_3^{-1} \mathcal{M}_i = T_3^{-1} \mathcal{M}_i T_4^{-1}$ and $T_1 \mathcal{M}'_i = T_1 \mathcal{M}'_i T_2$, $i = 1, 2$. Therefore, we obtain

$$\varphi(T_3^{-1} \mathcal{M}_i T_4^{-1}) \subseteq T_1 \mathcal{M}'_i T_2, \quad i = 1, 2.$$

Case 2. $\varphi(\mathcal{M}_1)$ is contained in a maximal set of type two. Similarly, there exist two invertible matrices T_1 and T_2 over \mathbb{D}' , and two invertible matrices $T_3^{-1} = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL_m(\mathbb{D})$ and $T_4^{-1} = \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix} \in GL_n(\mathbb{D})$, such that $\varphi(T_3^{-1} \mathcal{M}_i T_4^{-1}) \subseteq T_1 \mathcal{N}'_i T_2$ with $\varphi(T_3^{-1} \mathcal{N}_i T_4^{-1}) \subseteq T_1 \mathcal{M}'_i T_2$, $i = 1, 2$. \square

Theorem 3.4 Suppose \mathbb{D}, \mathbb{D}' are division rings and $m', n' \geq 2$ are integers. Let $\phi : \mathbb{D}^{2 \times 2} \rightarrow \mathbb{D}'^{m' \times n'}$ be a non-degenerate additive graph homomorphism. Then either there exist matrices $P \in GL_{m'}(\mathbb{D}')$ and $Q \in GL_{n'}(\mathbb{D}')$, and a nonzero ring homomorphism $\tau : \mathbb{D} \rightarrow \mathbb{D}'$, such that

$$\phi(X) = P \begin{pmatrix} X^\tau & 0 \\ 0 & 0 \end{pmatrix} Q, \quad X \in \mathbb{D}^{2 \times 2}; \quad (7)$$

or there exist matrices $P \in GL_{m'}(\mathbb{D}')$ and $Q \in GL_{n'}(\mathbb{D}')$, and a nonzero ring anti-homomorphism $\sigma : \mathbb{D} \rightarrow \mathbb{D}'$, such that

$$\phi(X) = P \begin{pmatrix} {}^t X^\sigma & 0 \\ 0 & 0 \end{pmatrix} Q, \quad X \in \mathbb{D}^{2 \times 2}. \quad (8)$$

Proof. By Lemma 3.3, we may assume with no loss of generality either $\phi(\mathcal{M}_i) \subseteq \mathcal{M}'_i$ with $\phi(\mathcal{N}_i) \subseteq \mathcal{N}'_i$ ($i = 1, 2$), or $\phi(\mathcal{M}_i) \subseteq \mathcal{N}'_i$ with $\phi(\mathcal{N}_i) \subseteq \mathcal{M}'_i$ ($i = 1, 2$). We prove this result only for the first case; the second case is similar. From now on we assume that

$$\phi(\mathcal{M}_i) \subseteq \mathcal{M}'_i \quad \text{with} \quad \phi(\mathcal{N}_i) \subseteq \mathcal{N}'_i, \quad i = 1, 2. \quad (9)$$

We show that ϕ is of the form (7) as follows. (For the second case, we can prove similarly ϕ is of the form (8).)

Let $E_{ij} = E_{ij}^{2 \times 2}$ and $E'_{ij} = E_{ij}^{m' \times n'}$. By (9) we have that $\phi(x E_{ij}) = x^{\sigma_{ij}} E'_{ij}$ for all $x \in \mathbb{D}$, $i, j = 1, 2$, where $\sigma_{ij} : \mathbb{D} \rightarrow \mathbb{D}'$ is an additive map with $0^{\sigma_{ij}} = 0$. Since ϕ is additive, we get

$$\phi \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} x_{11}^{\sigma_{11}} & x_{12}^{\sigma_{12}} \\ x_{21}^{\sigma_{21}} & x_{22}^{\sigma_{22}} \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}, \quad x_{ij} \in \mathbb{D}.$$

Without loss of generality, we assume that $1^{\sigma_{11}} = 1^{\sigma_{22}} = 1$.

Since $\begin{pmatrix} x & x \\ 1 & 1 \end{pmatrix} \sim 0$ and $\begin{pmatrix} 1 & x \\ 1 & x \end{pmatrix} \sim 0$ for all $x \in \mathbb{D}$, $\begin{pmatrix} x^{\sigma_{11}} & x^{\sigma_{12}} \\ 1^{\sigma_{21}} & 1 \end{pmatrix} \sim 0$ and $\begin{pmatrix} 1 & x^{\sigma_{12}} \\ 1^{\sigma_{21}} & x^{\sigma_{22}} \end{pmatrix} \sim 0$ for all $x \in \mathbb{D}$. It follows that $x^{\sigma_{11}} = x^{\sigma_{12}} 1^{\sigma_{21}}$ and $x^{\sigma_{22}} = 1^{\sigma_{21}} x^{\sigma_{12}}$ for all $x \in \mathbb{D}$. Therefore, $1 = 1^{\sigma_{21}} 1^{\sigma_{12}} = 1^{\sigma_{12}} 1^{\sigma_{21}}$ and hence $1^{\sigma_{12}} = (1^{\sigma_{21}})^{-1}$. Replacing the map ϕ by the map

$$X \mapsto \text{diag}(1, (1^{\sigma_{21}})^{-1}, 1, \dots, 1) \phi(X) \text{diag}(1, (1^{\sigma_{12}})^{-1}, 1, \dots, 1),$$

we have that $1^{\sigma_{ij}} = 1$ for $i, j = 1, 2$. Then $x^{\sigma_{11}} = x^{\sigma_{22}} = x^{\sigma_{12}}$ for all $x \in \mathbb{D}$. Write $\tau = \sigma_{11}$. Since $\begin{pmatrix} x & x \\ x & x \end{pmatrix} \sim 0$ for all $x \in \mathbb{D}^*$, $\begin{pmatrix} x^\tau & x^\tau \\ x^{\sigma_{21}} & x^\tau \end{pmatrix} \sim 0$ for all $x \in \mathbb{D}^*$, which implies that $\sigma_{21} = \tau$. Since $\begin{pmatrix} xy & x \\ y & 1 \end{pmatrix} \sim 0$ for all $x, y \in \mathbb{D}$, $\begin{pmatrix} (xy)^\tau & x^\tau \\ y^\tau & 1 \end{pmatrix} \sim 0$ for all $x, y \in \mathbb{D}$. Consequently $(xy)^\tau = x^\tau y^\tau$ for all $x, y \in \mathbb{D}$. Thus τ is a nonzero ring homomorphism, and hence (7) holds. \square

Corollary 3.5 *If $\phi : \mathbb{D}^{2 \times 2} \rightarrow \mathbb{D}'^{m' \times n'}$ ($m', n' \geq 2$) is a non-degenerate additive graph homomorphism, then ϕ is a distance preserving map.*

Theorem 3.6 *Suppose \mathbb{D}, \mathbb{D}' are division rings and $m, n, m', n' \geq 2$ are integers. Let $\phi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ be a non-degenerate additive graph homomorphism. Then either there exist matrices $P \in \mathbb{D}'^{m' \times m}$ and $Q \in \mathbb{D}^{n \times n'}$ with their ranks ≥ 2 , and a nonzero ring homomorphism $\tau : \mathbb{D} \rightarrow \mathbb{D}'$, such that*

$$\phi(X) = P X^\tau Q, \quad X \in \mathbb{D}^{m \times n}; \quad (10)$$

or there exist matrices $P \in \mathbb{D}'^{m' \times n}$ and $Q \in \mathbb{D}^{m \times n'}$ with their ranks ≥ 2 , and a nonzero ring anti-homomorphism $\sigma : \mathbb{D} \rightarrow \mathbb{D}'$, such that

$$\phi(X) = P^t X^\sigma Q, \quad X \in \mathbb{D}^{m \times n}. \quad (11)$$

Proof. By Lemma 3.3, we may assume with no loss of generality either $\phi(\mathcal{M}_i) \subseteq \mathcal{M}'_i$ with $\phi(\mathcal{N}_i) \subseteq \mathcal{N}'_i$ ($i = 1, 2$), or $\phi(\mathcal{M}_i) \subseteq \mathcal{N}'_i$ with $\phi(\mathcal{N}_i) \subseteq \mathcal{M}'_i$ ($i = 1, 2$). If the first case [resp. second case] happens, we can prove that ϕ is of the form (10) [resp. (11)]. We prove this theorem only for the first case; the second case is similar. From now on we assume that

$$\phi(\mathcal{M}_i) \subseteq \mathcal{M}'_i, \quad \phi(\mathcal{N}_i) \subseteq \mathcal{N}'_i, \quad i = 1, 2. \quad (12)$$

By Theorem 3.4 and its proof, we may assume with no loss of generality that

$$\phi \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X^\tau & 0 \\ 0 & 0 \end{pmatrix}, \quad X \in \mathbb{D}^{2 \times 2}, \quad (13)$$

where $\tau : \mathbb{D} \rightarrow \mathbb{D}'$ is a nonzero ring homomorphism.

Write $E_{ij} = E_{ij}^{m \times n}$ and $E'_{ij} = E'_{ij}^{m' \times n'}$. For $i \in \{1, 2\}$ and $j \in \{3, \dots, n\}$, we let

$$\psi_{ij} \begin{pmatrix} x_{1i} & x_{1j} \\ x_{2i} & x_{2j} \end{pmatrix} = \phi \begin{pmatrix} x_{1i} E_{1i} + x_{1j} E_{1j} + x_{2i} E_{2i} + x_{2j} E_{2j} \end{pmatrix}, \quad x_{1i}, x_{1j}, x_{2i}, x_{2j} \in \mathbb{D}.$$

Then $\psi_{ij} : \mathbb{D}^{2 \times 2} \rightarrow \mathbb{D}'^{m' \times n'}$ is an additive graph homomorphism. For $j \in \{3, \dots, n\}$, there exists some $k \in \{1, 2\}$ such that ψ_{kj} is non-degenerate. Otherwise, both ψ_{1j} and ψ_{2j} are degenerate, and Theorem 3.2 implies that $\psi_{1j}(\mathbb{D}^{2 \times 2})$ and $\psi_{2j}(\mathbb{D}^{2 \times 2})$ are two adjacent sets. By Corollary 2.3, (13) and Lemma 2.5, it is

easy to see that $\psi_{1j}(\mathbb{D}^{2 \times 2}) \subseteq \mathcal{N}'_1$ and $\psi_{2j}(\mathbb{D}^{2 \times 2}) \subseteq \mathcal{N}'_2$. It follows that $\phi(x_{1j}E_{1j} + x_{2j}E_{2j}) \in \mathcal{N}'_1 \cap \mathcal{N}'_2 = \{0\}$ for all $x_{1j}, x_{2j} \in \mathbb{D}$, a contradiction. Thus ψ_{kj} is non-degenerate for some $k \in \{1, 2\}$.

By Theorem 3.4 and (13), we can assume that

$$\psi_{kj} \begin{pmatrix} x_{1k} & x_{1j} \\ x_{2k} & x_{2j} \end{pmatrix} = \phi(x_{1k}E_{1k} + x_{1j}E_{1j} + x_{2k}E_{2k} + x_{2j}E_{2j}) = P_k \begin{pmatrix} \begin{pmatrix} x_{1k}^\mu & x_{1j}^\mu \\ x_{2k}^\mu & x_{2j}^\mu \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} Q_j, \quad (14)$$

for all $x_{1k}, x_{1j}, x_{2k}, x_{2j} \in \mathbb{D}$, where $P_k \in GL_{m'}(\mathbb{D}')$ is a diagonal matrix, $Q_j \in GL_{n'}(\mathbb{D}')$ and $\mu : \mathbb{D} \rightarrow \mathbb{D}'$ is a nonzero ring homomorphism. Let $q_{ji} = (q_{ji}^{(1)}, \dots, q_{ji}^{(n')})$ be the i -th row of Q_j and let $P_k = \text{diag}(p_{k1}, \dots, p_{km'})$. Then (14) and (13) imply that

$$\phi(xE_{1k}) = \begin{pmatrix} p_{k1}x^\mu q_{j1} \\ 0 \\ 0_{m'-2, n'} \end{pmatrix} = x^\tau E'_{1k}, \quad \phi(xE_{2k}) = \begin{pmatrix} 0 \\ p_{k2}x^\mu q_{j1} \\ 0_{m'-2, n'} \end{pmatrix} = x^\tau E'_{2k}, \quad x \in \mathbb{D}; \quad (15)$$

$$\phi(xE_{1j}) = \begin{pmatrix} p_{k1}x^\mu q_{j2} \\ 0 \\ 0_{m'-2, n'} \end{pmatrix}, \quad \phi(xE_{2j}) = \begin{pmatrix} 0 \\ p_{k2}x^\mu q_{j2} \\ 0_{m'-2, n'} \end{pmatrix}, \quad x \in \mathbb{D}. \quad (16)$$

By (15), we have $x^\tau = p_{k1}x^\mu q_{j1}^{(k)}$ and $x^\tau = p_{k2}x^\mu q_{j1}^{(k)}$ for all $x \in \mathbb{D}$. Thus $p_{k1} = p_{k2} = (q_{j1}^{(k)})^{-1}$.

Let e_i be the i -th row of $I_{m'}$, and let e'_i be the i -th row of $I_{n'}$. Write $q'_{j2} = (q_{j1}^{(k)})^{-1} q_{j2}$. Then (16) can be written as

$$\phi(xE_{1j}) = {}^t e_1 x^\tau q'_{j2}, \quad \phi(xE_{2j}) = {}^t e_2 x^\tau q'_{j2}, \quad x \in \mathbb{D}, \quad j = 3, \dots, n. \quad (17)$$

For $s \in \{1, 2\}$ and $i \in \{3, \dots, n\}$, we define

$$\psi'_{si} \begin{pmatrix} x_{s1} & x_{s2} \\ x_{i1} & x_{i2} \end{pmatrix} = \phi(x_{s1}E_{s1} + x_{s2}E_{s2} + x_{i1}E_{i1} + x_{i2}E_{i2}), \quad x_{s1}, x_{s2}, x_{i1}, x_{i2} \in \mathbb{D}.$$

Then $\psi'_{si} : \mathbb{D}^{2 \times 2} \rightarrow \mathbb{D}'^{m' \times n'}$ is an additive graph homomorphism. Similarly, there exists some $s \in \{1, 2\}$ such that ψ'_{si} is non-degenerate. Similar to the proof of (17), there is $0 \neq p'_{i2} \in \mathbb{D}'^{m' \times 1}$ such that

$$\phi(xE_{i1}) = p'_{i2} x^\tau e'_1, \quad \phi(xE_{i2}) = p'_{i2} x^\tau e'_2, \quad x \in \mathbb{D}, \quad i = 3, \dots, n. \quad (18)$$

Recall that ψ_{kj} is non-degenerate for some $k \in \{1, 2\}$. Put $i \in \{3, \dots, m\}$, $j \in \{3, \dots, n\}$ and $r \in \{1, 2\}$. We let

$$\theta_{ij}^{(r)} \begin{pmatrix} x_{rk} & x_{rj} \\ x_{ik} & x_{ij} \end{pmatrix} = \phi(x_{rk}E_{rk} + x_{rj}E_{rj} + x_{ik}E_{ik} + x_{ij}E_{ij}), \quad x_{rk}, x_{rj}, x_{ik}, x_{ij} \in \mathbb{D}.$$

Then $\theta_{ij}^{(r)} : \mathbb{D}^{2 \times 2} \rightarrow \mathbb{D}'^{m' \times n'}$ is an additive graph homomorphism. There exists some $r \in \{1, 2\}$ such that $\theta_{ij}^{(r)}$ is non-degenerate. Otherwise, both $\theta_{ij}^{(1)}$ and $\theta_{ij}^{(2)}$ are degenerate. By Theorem 3.2, we have $\theta_{ij}^{(1)}(\mathbb{D}^{2 \times 2}) \subseteq \mathcal{M}$, where \mathcal{M} is a maximal set in $\mathbb{D}'^{m' \times n'}$ containing 0. From (13) we get $\theta_{ij}^{(1)} \begin{pmatrix} x_{1k} & 0 \\ 0 & 0 \end{pmatrix} = x_{1k}^\tau E'_{1k}$ for all $x_{1k} \in \mathbb{D}$. Thus, Corollaries 2.2 and 2.3 imply that $\mathcal{M} = \mathcal{M}'_1$ or $\mathcal{M} = \mathcal{N}'_k$. Suppose that $\mathcal{M} = \mathcal{N}'_k$. Then (12) implies that $\theta_{ij}^{(1)} \begin{pmatrix} x_{1k} & x_{1j} \\ 0 & 0 \end{pmatrix} \in \mathcal{N}'_k \cap \mathcal{M}'_1 = \mathbb{D}' E'_{1k}$, and hence $\phi(E_{1j}) = a_1 E'_{1k}$ where $a_1 \in \mathbb{D}'^*$. By (15) and (16), it is easy to see that q_{j1} and q_{j2} are left linearly dependent, a contradiction to Q_j being invertible. Therefore, we must have $\mathcal{M} = \mathcal{M}'_1$ and hence $\theta_{ij}^{(1)}(\mathbb{D}^{2 \times 2}) \subseteq \mathcal{M}'_1$. Similarly,

$\theta_{ij}^{(2)}(\mathbb{D}^{2 \times 2}) \subseteq \mathcal{M}'_2$. It follows that $\phi(x_{ik}E_{ik} + x_{ij}E_{ij}) \in \mathcal{M}'_1 \cap \mathcal{M}'_2 = \{0\}$ for all $x_{ik}, x_{ij} \in \mathbb{D}$, a contradiction. Thus, $\theta_{ij}^{(r)}$ is non-degenerate for some $r \in \{1, 2\}$.

By Theorem 3.4 and (12), we have

$$\theta_{ij}^{(r)} \begin{pmatrix} x_{rk} & x_{rj} \\ x_{ik} & x_{ij} \end{pmatrix} = \phi(x_{rk}E_{rk} + x_{rj}E_{rj} + x_{ik}E_{ik} + x_{ij}E_{ij}) = S_i \begin{pmatrix} \begin{pmatrix} x_{rk}^\delta & x_{rj}^\delta \\ x_{ik}^\delta & x_{ij}^\delta \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} T_j, \quad (19)$$

for all $x_{rk}, x_{rj}, x_{ik}, x_{ij} \in \mathbb{D}$, where $S_i \in GL_{m'}(\mathbb{D}')$, $T_j \in GL_{n'}(\mathbb{D}')$ and $\delta : \mathbb{D} \rightarrow \mathbb{D}'$ is a nonzero ring homomorphism. Let $t_{ji} = (t_{ji}^{(1)}, \dots, t_{ji}^{(n')})$ be the i -th row of T_j , and let $S_i = (s_{i1}, \dots, s_{im'})$ where s_{ij} be the j -th column of S_i . Then (19) and (13) imply that

$$\phi(xE_{rk}) = s_{i1}x^\delta t_{j1} = x^\tau E'_{rk} = {}^t e_r x^\tau e'_k, \quad x \in \mathbb{D}; \quad (20)$$

$$\phi(xE_{rj}) = s_{i1}x^\delta t_{j2}, \quad x \in \mathbb{D};$$

$$\phi(xE_{ik}) = s_{i2}x^\delta t_{j1}, \quad x \in \mathbb{D};$$

$$\phi(xE_{ij}) = s_{i2}x^\delta t_{j2}, \quad x \in \mathbb{D}.$$

By (20), there are $a, b \in \mathbb{D}'^*$ such that $s_{i1} = {}^t e_r a$ and $t_{j1} = b e'_k$. Moreover, $x^\delta = a^{-1}x^\tau b^{-1}$ for all $x \in \mathbb{D}$. Since $1^\delta = 1^\tau = 1$, we get that $b^{-1} = a$. Hence

$$s_{i1} = {}^t e_r a, \quad at_{j1} = e'_k \quad \text{and} \quad x^\delta = a^{-1}x^\tau a, \quad x \in \mathbb{D}.$$

Therefore, for above two fixed $k, r \in \{1, 2\}$, we obtain that

$$\phi(xE_{rj}) = {}^t e_r x^\tau at_{j2}, \quad x \in \mathbb{D}, j = 3, \dots, n; \quad (21)$$

$$\phi(xE_{ik}) = s_{i2}a^{-1}x^\tau at_{j1} = s_{i2}a^{-1}x^\tau e'_k, \quad x \in \mathbb{D}, i = 3, \dots, m; \quad (22)$$

$$\phi(xE_{ij}) = s_{i2}a^{-1}x^\tau at_{j2}, \quad x \in \mathbb{D}, i = 3, \dots, m, j = 3, \dots, n. \quad (23)$$

By (21) and (17), we get $q'_{j2} = at_{j2}$. Thus (17) can be written as

$$\phi(xE_{1j}) = {}^t e_1 x^\tau at_{j2}, \quad \phi(xE_{2j}) = {}^t e_2 x^\tau at_{j2}, \quad x \in \mathbb{D}, j = 3, \dots, n. \quad (24)$$

By (18) and (22), we have $p'_{j2} = s_{i2}a^{-1}$. Hence (18) can be written as

$$\phi(xE_{i1}) = s_{i2}a^{-1}x^\tau e'_1, \quad \phi(xE_{i2}) = s_{i2}a^{-1}x^\tau e'_2, \quad x \in \mathbb{D}, i = 3, \dots, n. \quad (25)$$

Let $P = ({}^t e_1, {}^t e_2, s_{32}a^{-1}, s_{42}a^{-1}, \dots, s_{m2}a^{-1}) = \begin{pmatrix} I_2 & * \\ 0 & * \end{pmatrix} \in \mathbb{D}'^{m' \times m}$, and let

$$Q = \begin{pmatrix} e'_1 \\ e'_2 \\ at_{32} \\ at_{42} \\ \vdots \\ at_{n2} \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ * & * \end{pmatrix} \in \mathbb{D}'^{n \times n'}.$$

Then $\text{rank}(P) \geq 2$ and $\text{rank}(Q) \geq 2$. By (13) with (23)-(25), we obtain that

$$\phi(xE_{ij}) = Px^\tau E'_{ij}Q, \quad x \in \mathbb{D}, i = 1, \dots, m, j = 1, \dots, n.$$

Since ϕ is additive, we get (10). □

By Theorem 3.6 and Theorem 3.2, it is clear that Theorem 3.1 holds.

4 Graph homomorphisms on $n \times n$ matrices

In this section, we will expand [24, Theorem 4.1] (which is due to Šemrl) to the cases of two division rings and $n = 2$.

Theorem 4.1 *Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$, and let m', n', n be integers with $m', n' \geq n \geq 2$. Suppose that $\varphi : \mathbb{D}^{n \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism with $\varphi(0) = 0$, and there exists $A_0 \in \mathbb{D}^{n \times n}$ such that $\text{rank}(\varphi(A_0)) = n$. Then either there exist matrices $P \in GL_{m'}(\mathbb{D}')$ and $Q \in GL_{n'}(\mathbb{D}')$, a nonzero ring homomorphism $\tau : \mathbb{D} \rightarrow \mathbb{D}'$, and a matrix $L \in \mathbb{D}'^{n \times n}$ with the property that $I_n + X^\tau L \in GL_n(\mathbb{D}')$ for every $X \in \mathbb{D}^{n \times n}$, such that*

$$\varphi(X) = P \begin{pmatrix} (I_n + X^\tau L)^{-1} X^\tau & 0 \\ 0 & 0 \end{pmatrix} Q, \quad X \in \mathbb{D}^{n \times n}; \quad (26)$$

or there exist matrices $P \in GL_{m'}(\mathbb{D}')$ and $Q \in GL_{n'}(\mathbb{D}')$, a nonzero ring anti-homomorphism $\sigma : \mathbb{D} \rightarrow \mathbb{D}'$, and a matrix $L \in \mathbb{D}'^{n \times n}$ with the property that $I_n + L {}^t X^\sigma \in GL_n(\mathbb{D}')$ for every $X \in \mathbb{D}^{n \times n}$, such that

$$\varphi(X) = P \begin{pmatrix} {}^t X^\sigma (I_n + L {}^t X^\sigma)^{-1} & 0 \\ 0 & 0 \end{pmatrix} Q, \quad X \in \mathbb{D}^{n \times n}; \quad (27)$$

or $\varphi(\mathbb{D}_{\leq 1}^{n \times n})$ is an adjacent set. In particular, if τ [resp. σ] is surjective, then $L = 0$ and τ is a ring isomorphism [resp. σ is a ring anti-isomorphism].

By Theorem 4.1, it is easy to prove the following corollary.

Corollary 4.2 *Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$, and let m', n', n be integers with $m', n' \geq n \geq 2$. Suppose that $\varphi : \mathbb{D}^{n \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism, and there exist $A_0, B_0 \in \mathbb{D}^{n \times n}$ such that $\text{rank}(\varphi(B_0) - \varphi(A_0)) = n$. Then either there exist matrices $P \in GL_{m'}(\mathbb{D}')$ and $Q \in GL_{n'}(\mathbb{D}')$, a nonzero ring homomorphism $\tau : \mathbb{D} \rightarrow \mathbb{D}'$, and a matrix $L \in \mathbb{D}'^{n \times n}$ with the property that $I_n + X^\tau L \in GL_n(\mathbb{D}')$ for every $X \in \mathbb{D}^{n \times n}$, such that*

$$\varphi(X) = P \begin{pmatrix} (I_n + X^\tau L)^{-1} (X^\tau - A_0^\tau) & 0 \\ 0 & 0 \end{pmatrix} Q + \varphi(A_0), \quad X \in \mathbb{D}^{n \times n}; \quad (28)$$

or there exist matrices $P \in GL_{m'}(\mathbb{D}')$ and $Q \in GL_{n'}(\mathbb{D}')$, a nonzero ring anti-homomorphism $\sigma : \mathbb{D} \rightarrow \mathbb{D}'$, and a matrix $L \in \mathbb{D}'^{n \times n}$ with the property that $I_n + L {}^t X^\sigma \in GL_n(\mathbb{D}')$ for every $X \in \mathbb{D}^{n \times n}$, such that

$$\varphi(X) = P \begin{pmatrix} ({}^t X^\sigma - {}^t A_0^\sigma)(I_n + L {}^t X^\sigma)^{-1} & 0 \\ 0 & 0 \end{pmatrix} Q + \varphi(A_0), \quad X \in \mathbb{D}^{n \times n}; \quad (29)$$

or both $\varphi(\mathbb{B}_{A_0})$ and $\varphi(\mathbb{B}_{B_0})$ are adjacent sets. In particular, if τ [resp. σ] is surjective, then $L = 0$ and τ is a ring isomorphism [resp. σ is a ring anti-isomorphism].

Note that maps (26)-(29) are distance preserving maps (cf. [14, Examples 2.5 and 2.6]). We have:

Corollary 4.3 *Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$, and let m', n', n be integers with $m', n' \geq n \geq 2$. Suppose that $\varphi : \mathbb{D}^{n \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism, and there exist $A_0, B_0 \in \mathbb{D}^{n \times n}$ such that $\text{rank}(\varphi(B_0) - \varphi(A_0)) = n$. Then either φ is a distance preserving map, or both $\varphi(\mathbb{B}_{A_0})$ and $\varphi(\mathbb{B}_{B_0})$ are adjacent sets.*

In order to prove Theorem 4.1 we need the following knowledge and lemmas.

It is well-known that $\mathbb{D}^{m \times n}$ ($m, n \geq 2$) is a partially ordered set (poset) with partial order defined by $A \leq B$ if $\text{rank}(B - A) = \text{rank}(B) - \text{rank}(A)$. This partial order is also called *minus order* (or *minus partial order*) of $\mathbb{D}^{m \times n}$ (cf. [22]). In particular, we write $A < B \Leftrightarrow A \leq B$ with $A \neq B$.

For any $X \in \mathbb{D}^{m \times n}$, a g-inverse of X will be denoted by X^- and is understood as a matrix (over \mathbb{D}) for which $XX^-X = X$. Two matrices $A, B \in \mathbb{D}^{m \times n}$ are said to be *equivalent*, denoted by $A \cong B$, if B may be obtained from A by a finite sequence of elementary row and column operations.

Lemma 4.4 (cf. [22, Section 3.3]) *Let \mathbb{D} be a division ring and let $A, B \in \mathbb{D}^{m \times n}$. Then the following results are equivalent:*

- (a) $A \leq B$;
- (b) *there are g-inverses G_1 and G_2 of A such that $AG_1 = BG_1$ and $G_2A = G_2B$;*
- (c) *there are $P \in GL_m(\mathbb{D})$ and $Q \in GL_n(\mathbb{D})$ such that $B = P\text{diag}(I_{r+s}, 0)Q$ and $A = P\text{diag}(I_r, 0)Q$;*
- (d) $PAQ \leq PBQ$, where $P \in GL_m(\mathbb{D})$ and $Q \in GL_n(\mathbb{D})$.

Lemma 4.5 (see [22, Theorem 3.6.5]) *Let $A, B \in \mathbb{D}^{n \times n}$ with $B^2 = B$. Then $A \leq B$ if and only if $A = A^2 = AB = BA$.*

Lemma 4.6 *Let $A, B, C \in \mathbb{D}^{m \times n}$ and $d(A, B) = d(B, C) - d(A, C)$. Suppose that $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism and $d(B, C) = d(\varphi(B), \varphi(C))$. Then $d(\varphi(A), \varphi(B)) = d(\varphi(B), \varphi(C)) - d(\varphi(A), \varphi(C))$.*

Proof. We have $d(\varphi(A), \varphi(B)) \leq d(A, B) = d(B, C) - d(A, C) = d(\varphi(B), \varphi(C)) - d(A, C) \leq d(\varphi(B), \varphi(C)) - d(\varphi(A), \varphi(C))$. Hence

$$d(\varphi(A), \varphi(B)) \leq d(\varphi(B), \varphi(C)) - d(\varphi(A), \varphi(C)).$$

On the other hand, it is clear that $d(\varphi(A), \varphi(B)) \geq d(\varphi(B), \varphi(C)) - d(\varphi(A), \varphi(C))$. Thus $d(\varphi(A), \varphi(B)) = d(\varphi(B), \varphi(C)) - d(\varphi(A), \varphi(C))$. \square

Corollary 4.7 *Let $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ be a graph homomorphism with $\varphi(0) = 0$, and let $A, B \in \mathbb{D}^{m \times n}$. If $A \leq B$ and $\text{rank}(\varphi(B)) = \text{rank}(B)$, then $\varphi(A) \leq \varphi(B)$ and $\text{rank}(\varphi(A)) = \text{rank}(A)$.*

Proof. Assume that $A \leq B$ and $\text{rank}(\varphi(B)) = \text{rank}(B)$. Then $d(A, B) = d(B, 0) - d(A, 0)$ and $d(B, 0) = d(\varphi(B), \varphi(0))$. By Lemma 4.6, $d(\varphi(A), \varphi(B)) = d(\varphi(B), \varphi(0)) - d(\varphi(A), \varphi(0))$, hence $\varphi(A) \leq \varphi(B)$. It follows that $\text{rank}(\varphi(A)) = d(\varphi(A), \varphi(0)) = d(\varphi(B), \varphi(0)) - d(\varphi(A), \varphi(B)) \geq \text{rank}(B) - d(A, B) = \text{rank}(A)$. Since $\text{rank}(\varphi(A)) \leq \text{rank}(A)$, we obtain $\text{rank}(\varphi(A)) = \text{rank}(A)$. \square

Lemma 4.8 *Let \mathbb{D}, \mathbb{D}' be division rings, and let n, m', n' be integers with $m', n' \geq n \geq 2$. Suppose that $\varphi : \mathbb{D}^{n \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism with $\varphi(0) = 0$, and there exists $A_0 \in \mathbb{D}^{n \times n}$ such that $d(\varphi(A_0), 0) = n$. Assume that \mathcal{M} is a maximal set in $\mathbb{D}^{n \times n}$ containing 0 and $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ where \mathcal{M}' is a maximal set in $\mathbb{D}'^{m' \times n'}$. Then $\varphi(\mathcal{M})$ is not contained in any line in $AG(\mathcal{M}')$, and \mathcal{M}' is the unique maximal set containing $\varphi(\mathcal{M})$ in $\mathbb{D}'^{m' \times n'}$.*

Proof. Without loss of generality, we assume that \mathcal{M} is a maximal set of type two, and \mathcal{M}' is a maximal set of type one. By Lemma 1.1, there are invertible matrices P_1, P_2 such that $\mathcal{M} = \mathcal{N}_1 P_1$ and $\mathcal{M}' = P_2 \mathcal{M}'_1$. Replacing φ by the map $X \mapsto P_2^{-1} \varphi(X P_1)$, we have $\mathcal{M} = \mathcal{N}_1$, $\mathcal{M}' = \mathcal{M}'_1$ and $\varphi(\mathcal{N}_1) \subseteq \mathcal{M}'_1$.

Suppose $\varphi(\mathcal{N}_1)$ is contained in a line ℓ in $AG(\mathcal{M}'_1)$. We show a contradiction as follows. By the parametric equation of a line, there exists an invertible matrix Q such that $\ell = (\mathbb{D}' E_{11} + B)Q$, where $E_{11} = E_{11}^{m' \times n'}$ and $B \in \mathcal{M}'_1$. Since $0 \in \ell$, we can assume that $B = 0$ and $\ell = \mathbb{D}' E_{11} Q$. Replacing φ by the map $X \mapsto \varphi(X) Q^{-1}$, we have $\varphi(0) = 0$ and $\varphi(\mathcal{N}_1) \subseteq \ell = \mathbb{D}' E_{11}$. By the conditions, there is an $A_0 \in \mathbb{D}^{n \times n}$ such that $d(\varphi(A_0), 0) = n = d(A_0, 0)$. Let $A_0 = (B_1, B_2)$ where $B_1 \in \mathbb{D}_2^{n \times 2}$ and $B_2 \in \mathbb{D}_{n-2}^{n \times (n-2)}$. When $n = 2$, B_2 is absent. Put $A_1 = (B_1, 0) \in \mathbb{D}_2^{n \times n}$. Then $A_1 \leq A_0$. Using Corollary 4.7, we get $\text{rank}(\varphi(A_1)) = 2$. Clearly, there are two distinct points $Y_1, Y_2 \in \mathcal{N}_1$ such that $A_1 \sim Y_i$, $i = 1, 2$. Since $\varphi(Y_i) \subseteq \ell$, $\varphi(A_1)$ are adjacent to two distinct points in $\mathbb{D}' E_{11}$. By Lemma 2.1, $\varphi(A_1) \in \mathcal{M}'_1$ or $\varphi(A_1) \in \mathcal{N}'_1$, and hence $\text{rank}(\varphi(A_1)) \leq 1$, a contradiction.

Thus $\varphi(\mathcal{N}_1)$ is not contained in any line of $AG(\mathcal{M}'_1)$. By Corollary 2.3 and Lemma 2.5, it is easy to see that \mathcal{M}' is the unique maximal set containing $\varphi(\mathcal{M})$. \square

Lemma 4.9 (cf. [24]) *Let \mathbb{D}, \mathbb{D}' be division rings, and let m', n', n be integers with $m', n' \geq n \geq 2$. Suppose that $\varphi : \mathbb{D}^{n \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism with $\varphi(0) = 0$ and $\text{rank}(\varphi(I_n)) = n$. Then there exist invertible matrices $P \in GL_{m'}(\mathbb{D}')$ and $Q \in GL_{n'}(\mathbb{D}')$ such that*

$$\varphi(\text{diag}(I_r, 0)) = P \text{diag}(I_r, 0_{m'-r, n'-r}) Q, \quad r = 1, \dots, n. \quad (30)$$

Proof. Since $\text{diag}(I_{n-1}, 0) < I_n$ and $\text{rank}(\varphi(I_n)) = n$, from Corollary 4.7 we get that

$$\text{rank}(\varphi(\text{diag}(I_{n-1}, 0))) = n - 1 \quad \text{and} \quad \varphi(\text{diag}(I_{n-1}, 0)) < \varphi(I_n).$$

Using Lemma 4.4(c), there exist $Q_1 \in GL_{m'}(\mathbb{D}')$ and $Q_2 \in GL_{n'}(\mathbb{D}')$ such that $\varphi(I_n) = Q_1 \text{diag}(I_n, 0) Q_2$ and $\varphi(\text{diag}(I_{n-1}, 0)) = Q_1 \text{diag}(I_{n-1}, 0) Q_2$. Replacing φ by the map $X \mapsto Q_1^{-1} \varphi(X) Q_2^{-1}$, we have that $\varphi(I_n) = \text{diag}(I_n, 0)$ and $\varphi(\text{diag}(I_{n-1}, 0)) = \text{diag}(I_{n-1}, 0_{m'-n+1, n'-n+1})$.

Now, let $n \geq 3$. Since $\text{diag}(I_{n-2}, 0) < \text{diag}(I_{n-1}, 0)$, Corollary 4.7 implies that $\text{rank}(\varphi(\text{diag}(I_{n-2}, 0))) = n - 2$ and

$$\varphi(\text{diag}(I_{n-2}, 0)) < \text{diag}(I_{n-1}, 0_{m'-n+1, n'-n+1}). \quad (31)$$

Write $\varphi(\text{diag}(I_{n-2}, 0)) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where $A_{11} \in \mathbb{D}'^{(n-1) \times (n-1)}$. By (31), Lemma 4.5 and multiplications of matrices, it is easy to see that $A_{12} = 0$, $A_{21} = 0$ and A_{11} is an idempotent matrix. (Note that when $m' \neq n'$ and using Lemma 4.5, we can get two suitable square matrices by adding some zero elements on $\varphi(\text{diag}(I_{n-2}, 0))$ and $\text{diag}(I_{n-1}, 0_{m'-n+1, n'-n+1})$.) Since $\varphi(\text{diag}(I_{n-2}, 0)) \sim \text{diag}(I_{n-1}, 0_{m'-n+1, n'-n+1})$, we have $A_{22} = 0$. Hence $\varphi(\text{diag}(I_{n-2}, 0)) = \text{diag}(A_{11}, 0_{m'-n+1, n'-n+1})$, where A_{11} is an idempotent matrix of rank $n - 2$. There is $T \in GL_{n-1}(\mathbb{D}')$ such that $A_{11} = T^{-1} \text{diag}(I_{n-2}, 0) T$. Let $Q_3 = \text{diag}(T^{-1}, I_{m'-n+1})$ and $Q_4 = \text{diag}(T, I_{n'-n+1})$. Replacing φ by the map $X \mapsto Q_3^{-1} \varphi(X) Q_4^{-1}$, we get that $\varphi(I_n) = \text{diag}(I_n, 0_{m'-n, n'-n})$, $\varphi(\text{diag}(I_{n-1}, 0)) = \text{diag}(I_{n-1}, 0_{m'-n+1, n'-n+1})$ and $\varphi(\text{diag}(I_{n-2}, 0)) = \text{diag}(I_{n-2}, 0_{m'-n+2, n'-n+2})$.

Similarly, after by some transformations of the form $X \mapsto P' \varphi(X) Q'$, we can get $\varphi(\text{diag}(I_r, 0)) = \text{diag}(I_r, 0_{m'-r, n'-r})$, $r = 1, \dots, n$. Thus (30) holds. \square

A graph homomorphism $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ with $\varphi(0) = 0$ is called to satisfy Condition (I), if for any two maximal sets \mathcal{M} and \mathcal{N} of different types in $\mathbb{D}^{m \times n}$ with $0 \in \mathcal{M} \cap \mathcal{N}$, there are two maximal sets \mathcal{M}' and \mathcal{N}' of different types in $\mathbb{D}'^{m' \times n'}$, such that $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ and $\varphi(\mathcal{N}) \subseteq \mathcal{N}'$.

Lemma 4.10 *Let \mathbb{D}, \mathbb{D}' be division rings, and let m', n', n be integers with $m', n' \geq n \geq 2$. Suppose that $\varphi : \mathbb{D}^{n \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism with $\varphi(0) = 0$, and there exists $A_0 \in \mathbb{D}^{n \times n}$ such that $\text{rank}(\varphi(A_0)) = n$. Assume further that φ satisfies the Condition (I). Then*

$$\dim(\varphi(\mathcal{M}_1)) = \dim(\varphi(\mathcal{N}_1)) = n. \quad (32)$$

Proof. By $\varphi(0) = 0$ and (1), A_0 is invertible. Replacing φ by the map $X \mapsto \varphi(A_0^{-1}X)$, we can assume that $A_0 = I_n$. By Lemma 4.9, there exist invertible matrices $P \in GL_{m'}(\mathbb{D}')$ and $Q \in GL_{n'}(\mathbb{D}')$ such that $\varphi(\text{diag}(I_r, 0)) = P \text{diag}(I_r, 0) Q$, $r = 1, \dots, n$. Replacing φ by the map $X \mapsto P^{-1}\varphi(X)Q^{-1}$, we get that

$$\varphi(\text{diag}(I_r, 0)) = \text{diag}(I_r, 0_{m'-r, n'-r}), \quad r = 1, \dots, n. \quad (33)$$

Write $E_{ij} = E_{ij}^{n \times n}$ and $E'_{ij} = E_{ij}^{m' \times n'}$. Note that $\varphi(E_{11}) = E'_{11}$. Since φ satisfies the Condition (I), from Corollary 2.2 we have either $\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1$ with $\varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1$, or $\varphi(\mathcal{M}_1) \subseteq \mathcal{N}'_1$ with $\varphi(\mathcal{N}_1) \subseteq \mathcal{M}'_1$. We prove (32) only for the first case; the second case is similar. From now on we assume that

$$\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1 \quad \text{and} \quad \varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1. \quad (34)$$

For any $x \in \mathbb{D}^*$, by $\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1$, we can assume that $\varphi(E_{11} + xE_{12}) = \begin{pmatrix} 1^* & x^{\sigma_1} & 0^* \\ 0 & 0 & 0 \end{pmatrix}$. Since $E_{11} + xE_{12} \sim E_{11} + E_{22}$, it follows from (33) that $\text{rank} \begin{pmatrix} 1^* & -1 & x^{\sigma_1} & 0^* \\ 0 & -1 & 0 & 0 \end{pmatrix} = 1$. Thus $1^* = 1$ and $0^* = 0$. Therefore,

$$\varphi(E_{11} + xE_{12}) = E'_{11} + x^{\sigma_1} E'_{12}, \quad x \in \mathbb{D}, \quad (35)$$

where $\sigma_1 : \mathbb{D} \rightarrow \mathbb{D}'$ is an injective map with $0^{\sigma_1} = 0$. Similarly, we have

$$\varphi(E_{11} + xE_{21}) = E'_{11} + x^{\mu_1} E'_{21}, \quad x \in \mathbb{D}, \quad (36)$$

where $\mu_1 : \mathbb{D} \rightarrow \mathbb{D}'$ is an injective map with $0^{\mu_1} = 0$.

For $2 \leq k \leq n$ and $x_i, y_i \in \mathbb{D}$ ($i = 2, \dots, k$), we have $E_{11} + x_2E_{12} + \dots + x_kE_{1k} < \text{diag}(I_k, 0)$ and $E_{11} + y_2E_{21} + \dots + y_kE_{k1} < \text{diag}(I_k, 0)$. Thus Corollary 4.7 and (33) imply that $\varphi(E_{11} + x_2E_{12} + \dots + x_kE_{1k}) < \text{diag}(I_k, 0_{m'-k, n'-k})$ and $\varphi(E_{11} + y_2E_{21} + \dots + y_kE_{k1}) < \text{diag}(I_k, 0_{m'-k, n'-k})$. Using (34), it is easy to see that

$$\varphi(E_{11} + x_2E_{12} + \dots + x_kE_{1k}) = E'_{11} + x_2^* E'_{12} + \dots + x_k^* E'_{1k}, \quad 2 \leq k \leq n; \quad (37)$$

$$\varphi(E_{11} + y_2E_{21} + \dots + y_kE_{k1}) = E'_{11} + y_2^* E'_{21} + \dots + y_k^* E'_{k1}, \quad 2 \leq k \leq n, \quad (38)$$

where $x_j^*, y_j^* \in \mathbb{D}'$, $j = 2, \dots, k$. Moreover, by $\varphi(E_{11}) = E'_{11}$ and the adjacency, $(x_2, \dots, x_k) \neq 0$ [resp. $(y_2, \dots, y_k) \neq 0$] if and only if $(x_2^*, \dots, x_k^*) \neq 0$ [resp. $(y_2^*, \dots, y_k^*) \neq 0$].

For any $1 \leq r \leq n-2$ and $2 \leq k \leq n-r$, we let $A_k = E_{11}^{k \times k} + x_2E_{12}^{k \times k} + \dots + x_kE_{1k}^{k \times k} = \begin{pmatrix} 1 & \alpha_k \\ 0 & 0 \end{pmatrix}$, where $x_i \in \mathbb{D}$ and $\alpha_k := (x_2, \dots, x_k) \neq 0$. Then A_k is a $k \times k$ idempotent matrices of rank one. By Corollary 4.7 and (33), we have that

$$\begin{pmatrix} I_r & \\ & 0 \end{pmatrix} < \varphi \begin{pmatrix} I_r & & \\ & A_k & \\ & & 0 \end{pmatrix} < \begin{pmatrix} I_{r+k} & \\ & 0 \end{pmatrix},$$

and $\text{rank}(\varphi(\text{diag}(I_r, A_k, 0))) = r+1$. Write $\varphi(\text{diag}(I_r, A_k, 0)) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where $A_{11} \in \mathbb{D}'^{(r+k) \times (r+k)}$. By (33), $\varphi(\text{diag}(I_r, A_k, 0))$ is adjacent with both $\text{diag}(I_{r+1}, 0)$ and $\text{diag}(I_r, 0)$. It follows from Lemma 2.1 that $A_{22} = 0$. By $\varphi(\text{diag}(I_r, A_k, 0)) < \text{diag}(I_{r+k}, 0)$ and Lemma 4.5, we get that $A_{12} = 0, A_{21} = 0$

and A_{11} is an idempotent matrix of rank $r + 1$. (Note that when $m' \neq n'$ and using Lemma 4.5, we can get two suitable square matrices by adding some zero elements on $\varphi(\text{diag}(I_r, A_k, 0))$ and $\text{diag}(I_{r+k}, 0)$.) Thus $\varphi(\text{diag}(I_r, A_k, 0)) = \text{diag}(A_{11}, 0)$.

Write $A_{11} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ where $B_{11} \in \mathbb{D}'^{r \times r}$ and $B_{22} \in \mathbb{D}'^{k \times k}$. Using $\text{diag}(I_r, 0_k) < A_{11}$ and Lemma 4.5, we have similarly that $B_{12} = 0$ and $B_{21} = 0$. By $B_{22} \neq 0$ and $\text{diag}(I_r, 0_k) \sim A_{11}$, it is clear that $B_{11} = I_r$. Therefore, we obtain

$$\varphi(\text{diag}(I_r, A_k, 0)) = \text{diag}(I_r, B_{22}, 0),$$

where $B_{22} \in \mathbb{D}'^{k \times k}$ is an idempotent matrix of rank one.

Since $\text{diag}(I_r, A_k, 0) \sim \text{diag}(I_{r+1}, 0)$, $\text{diag}(I_r, B_{22}, 0) \sim \text{diag}(I_{r+1}, 0)$, and hence $B_{22} \sim E_{11}^{k \times k}$. By Lemma 2.1, we obtain either $B_{22} = \begin{pmatrix} 1 & \alpha_k^* \\ 0 & 0 \end{pmatrix}$ or $B_{22} = \begin{pmatrix} 1 & 0 \\ \alpha_k^* & 0 \end{pmatrix}$, where $0 \neq \alpha_k^* \in \mathbb{D}'^{k-1}$. Since $E_{11} + E_{r+1,1} \leq \text{diag}(I_r, A_k, 0)$, Corollary 4.7 implies that $\varphi(E_{11} + E_{r+1,1}) \leq \varphi(\text{diag}(I_r, A_k, 0))$. From (38) we must have $B_{22} = \begin{pmatrix} 1 & \alpha_k^* \\ 0 & 0 \end{pmatrix}$. Therefore, for any $0 \neq \alpha_k \in \mathbb{D}^{k-1}$, there is $0 \neq \alpha_k^* \in \mathbb{D}'^{k-1}$ such that

$$\varphi \begin{pmatrix} I_r & & \\ & \begin{pmatrix} 1 & \alpha_k \\ 0 & 0 \end{pmatrix} & \\ & & 0 \end{pmatrix} = \begin{pmatrix} I_r & & \\ & \begin{pmatrix} 1 & \alpha_k^* \\ 0 & 0 \end{pmatrix} & \\ & & 0 \end{pmatrix}, \quad 1 \leq r \leq n-2, \quad 2 \leq k \leq n-r. \quad (39)$$

Similarly, for any $0 \neq \beta_k \in {}^{k-1}\mathbb{D}$, there is $0 \neq \beta_k^* \in {}^{k-1}\mathbb{D}'$ such that

$$\varphi \begin{pmatrix} I_r & & \\ & \begin{pmatrix} 1 & 0 \\ \beta_k & 0 \end{pmatrix} & \\ & & 0 \end{pmatrix} = \begin{pmatrix} I_r & & \\ & \begin{pmatrix} 1 & 0 \\ \beta_k^* & 0 \end{pmatrix} & \\ & & 0 \end{pmatrix}, \quad 1 \leq r \leq n-2, \quad 2 \leq k \leq n-r. \quad (40)$$

For any $2 \leq k \leq n$, by (37) we write

$$\varphi \left(E_{11} + \sum_{j=2}^k E_{1j} \right) = E'_{11} + \sum_{j=2}^k a_{1j}^{(k)} E'_{1j}, \quad (41)$$

where $a_{1j}^{(k)} \in \mathbb{D}'$ and $(a_{12}^{(k)}, \dots, a_{1k}^{(k)}) \neq (0, \dots, 0)$. From (35) we have $a_{12}^{(2)} \neq 0$.

We prove $a_{1k}^{(k)} \neq 0$ ($3 \leq k \leq n$) as follows. For any $3 \leq k \leq n$ and $2 \leq r < k$, by (39), we let

$$\varphi \left(E_{11} + \dots + E_{rr} + \sum_{j=r+1}^k E_{rj} \right) = E'_{11} + \dots + E'_{rr} + \sum_{j=r+1}^k a_{rj}^{(k)} E'_{rj},$$

where $a_{rj}^{(k)} \in \mathbb{D}'$ and $(a_{r,r+1}^{(k)}, \dots, a_{rk}^{(k)}) \neq (0, \dots, 0)$.

By (33) and the adjacency, when $r = k-1$ we have $a_{k-1,k}^{(k)} \neq 0$. We show $a_{rk}^{(k)} \neq 0$, $r = 2, \dots, k-2$.

Since $\varphi(E_{11} + \dots + E_{k-1,k-1} + E_{k-1,k}) \sim \varphi(E_{11} + \dots + E_{k-2,k-2} + E_{k-2,k-1} + E_{k-2,k})$, we get

$$E'_{11} + \dots + E'_{k-1,k-1} + a_{k-1,k}^{(k)} E'_{k-1,k} \sim E'_{11} + \dots + E'_{k-2,k-2} + a_{k-2,k-1}^{(k)} E'_{k-2,k-1} + a_{k-2,k}^{(k)} E'_{k-2,k},$$

which implies that $\text{rank} \begin{pmatrix} a_{k-2,k-1}^{(k)} & a_{k-2,k}^{(k)} \\ 1 & a_{k-1,k}^{(k)} \end{pmatrix} = 1$. By $(a_{k-2,k-1}^{(k)}, a_{k-2,k}^{(k)}) \neq (0, 0)$ and $a_{k-1,k}^{(k)} \neq 0$, we have $a_{k-2,k}^{(k)} \neq 0$.

Since $\varphi(E_{11} + \cdots + E_{k-2,k-2} + E_{k-2,k-1} + E_{k-2,k}) \sim \varphi(E_{11} + \cdots + E_{k-3,k-3} + \sum_{j=k-2}^k E_{k-3,j})$, one has

$$E'_{11} + \cdots + E'_{k-2,k-2} + a_{k-2,k-1}^{(k)} E'_{k-2,k-1} + a_{k-2,k}^{(k)} E'_{k-2,k} \sim E'_{11} + \cdots + E'_{k-3,k-3} + \sum_{j=k-2}^k a_{k-3,j}^{(k)} E'_{k-3,j}.$$

Similarly, we have $a_{k-3,k}^{(k)} \neq 0$. In the similar way, we can prove that $a_{k-4,k}^{(k)} \neq 0, \dots, a_{2k}^{(k)} \neq 0$.

Since $\varphi(E_{11} + \sum_{j=2}^k E_{1j}) \sim \varphi(E_{11} + E_{22} + \sum_{j=3}^k E_{2j})$, $E'_{11} + \sum_{j=2}^k a_{1j}^{(k)} E'_{1j} \sim E'_{11} + E'_{22} + \sum_{j=3}^k a_{2j}^{(k)} E'_{2j}$. It follows that

$$\text{rank} \begin{pmatrix} a_{12}^{(k)} & a_{13}^{(k)} & \cdots & a_{1k}^{(k)} \\ 1 & a_{23}^{(k)} & \cdots & a_{2k}^{(k)} \end{pmatrix} = 1.$$

By $(a_{12}^{(k)}, \dots, a_{1k}^{(k)}) \neq (0, \dots, 0)$ and $a_{2k}^{(k)} \neq 0$, we must have $a_{1k}^{(k)} \neq 0$. Therefore, we have proved that

$$a_{1k}^{(k)} \neq 0, \quad k = 2, \dots, n.$$

Let $\gamma_1 = E_{11}$, $\gamma_k = E_{11} + \sum_{j=2}^k E_{1j}$, $\gamma_1^* = E'_{11}$ and $\gamma_k^* = E'_{11} + \sum_{j=2}^k a_{1j}^{(k)} E'_{1j}$, $k = 2, \dots, n$. Then $\gamma_i \in \mathcal{M}_1$, $\gamma_i^* \in \mathcal{M}'_1$, $\gamma_1, \dots, \gamma_n$ are left linearly independent over \mathbb{D} , and $\gamma_1^*, \dots, \gamma_n^*$ are left linearly independent over \mathbb{D}' . By (41) we have $\varphi(\gamma_k) = \gamma_k^*$, $k = 1, \dots, n$, and hence $\dim(\varphi(\mathcal{M}_1)) \geq n$. By Lemma 2.7, we get $\dim(\varphi(\mathcal{M}_1)) \leq n$. Hence $\dim(\varphi(\mathcal{M}_1)) = n$.

Similarly, we can prove $\dim(\varphi(\mathcal{N}_1)) = n$. Hence $\dim(\varphi(\mathcal{M}_1)) = \dim(\varphi(\mathcal{N}_1)) = n$. \square

Lemma 4.11 *Let \mathbb{D}, \mathbb{D}' be division rings, and let m', n', n be integers with $m', n' \geq n \geq 2$. Suppose that $\varphi : \mathbb{D}^{n \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism with $\varphi(0) = 0$, and there exists $A_0 \in \mathbb{D}^{n \times n}$ such that $\text{rank}(\varphi(A_0)) = n$. Assume further that φ does not satisfies the Condition (I). Then there exist two maximal sets \mathcal{M}' and \mathcal{N}' of different types in $\mathbb{D}'^{m' \times n'}$ such that $\varphi(\mathbb{D}_{\leq 1}^{n \times n}) \subseteq \mathcal{M}' \cup \mathcal{N}'$. Moreover, if $|\mathbb{D}| \geq 4$, then $\varphi(\mathbb{D}_{\leq 1}^{n \times n})$ is an adjacent set.*

Proof. Step 1. Since φ does not satisfies the Condition (I), there are two maximal sets \mathcal{M} and \mathcal{R} of different types in $\mathbb{D}^{m \times n}$ with $0 \in \mathcal{M} \cap \mathcal{R}$, such that $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ and $\varphi(\mathcal{R}) \subseteq \mathcal{R}'$, where \mathcal{M}' and \mathcal{R}' are two maximal sets of the same type containing 0 in $\mathbb{D}'^{m' \times n'}$. Since $|\mathcal{M} \cap \mathcal{R}| \geq 2$, $|\mathcal{M}' \cap \mathcal{R}'| \geq 2$. It follows from Lemma 2.3 that $\mathcal{M}' = \mathcal{R}'$. Thus

$$\varphi(\mathcal{M}) \subseteq \mathcal{M}' \text{ and } \varphi(\mathcal{R}) \subseteq \mathcal{M}'. \quad (42)$$

Without loss of generality, we assume that both \mathcal{M} and \mathcal{M}' are of type one. Hence \mathcal{R} is of type two. By Lemma 2.6(b), every non-degenerate graph homomorphism satisfies the Condition (I). Thus φ is degenerate.

We prove that there exist a type two maximal set \mathcal{N}' such that $\varphi(\mathbb{D}_{\leq 1}^{n \times n}) \subseteq \mathcal{M}' \cup \mathcal{N}'$ as follows. Suppose $\varphi(\mathbb{D}_{\leq 1}^{n \times n})$ is an adjacent set. By (42) and Lemma 4.8, it is easy to see that $\varphi(\mathbb{D}_{\leq 1}^{n \times n}) \subseteq \mathcal{M}'$. Now, suppose $\varphi(\mathbb{D}_{\leq 1}^{n \times n})$ is not an adjacent set. Then there exists a type one maximal set \mathcal{N} containing

0 in $\mathbb{D}^{n \times n}$, such that $\mathcal{M} \neq \mathcal{N}$, $\varphi(\mathcal{N}) \subseteq \mathcal{N}'$ and $\mathcal{N}' \neq \mathcal{M}'$, where \mathcal{N}' is a maximal set containing 0 in $\mathbb{D}^{m' \times n'}$. Otherwise, for any type one maximal set \mathcal{N} containing 0 in $\mathbb{D}^{n \times n}$, we have $\varphi(\mathcal{N}) \subseteq \mathcal{M}'$. Since every matrix in $\mathbb{D}_{\leq 1}^{n \times n}$ is contained in some type one maximal set containing 0, we get $\varphi(\mathbb{D}_{\leq 1}^{n \times n}) \subseteq \mathcal{M}'$, a contradiction to assumption. By Corollary 2.3, we have $|\mathcal{N} \cap \mathcal{R}| \geq 2$, hence $|\mathcal{N}' \cap \mathcal{M}'| \geq 2$. It follows that \mathcal{N}' is of type two. Let \mathcal{S} be any type two maximal set containing 0 in $\mathbb{D}^{n \times n}$, and let $\varphi(\mathcal{S}) \subseteq \mathcal{S}'$ where \mathcal{S}' is a maximal set containing 0 in $\mathbb{D}^{m' \times n'}$. By Corollary 2.3, we have $|\mathcal{S} \cap \mathcal{M}| \geq 2$ and $|\mathcal{S} \cap \mathcal{N}| \geq 2$, hence $|\mathcal{S}' \cap \mathcal{M}'| \geq 2$ and $|\mathcal{S}' \cap \mathcal{N}'| \geq 2$. Applying Corollary 2.3 again, we get either $\mathcal{S}' = \mathcal{M}'$ or $\mathcal{S}' = \mathcal{N}'$, and hence $\mathcal{S}' \subseteq \mathcal{M}' \cup \mathcal{N}'$. Consequently, $\varphi(\mathcal{S}) \subseteq \mathcal{M}' \cup \mathcal{N}'$ for any type two maximal set \mathcal{S} containing 0 in $\mathbb{D}^{n \times n}$. Since every matrix in $\mathbb{D}_{\leq 1}^{n \times n}$ is contained in some type two maximal set containing 0, we get $\varphi(\mathbb{D}_{\leq 1}^{n \times n}) \subseteq \mathcal{M}' \cup \mathcal{N}'$. Therefore, we always have

$$\varphi(\mathbb{D}_{\leq 1}^{n \times n}) \subseteq \mathcal{M}' \cup \mathcal{N}'. \quad (43)$$

Step 2. From now on we assume $|\mathbb{D}| \geq 4$. In this step, we will prove that $\varphi(\mathbb{D}_{\leq 1}^{n \times n})$ is an adjacent set.

By Lemma 2.4(i), there are two invertible matrices P_1 and Q_1 over \mathbb{D} , such that $\mathcal{M} = P_1 \mathcal{M}_1 Q_1$ and $\mathcal{R} = P_1 \mathcal{N}_1 Q_1$. Also, there is an invertible matrix P_2 over \mathbb{D}' , such that $\mathcal{M}' = P_2 \mathcal{M}'_1$. Replacing φ by the map $X \mapsto P_2^{-1} \varphi(P_1 X Q_1)$, (42) becomes

$$\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1 \quad \text{and} \quad \varphi(\mathcal{N}_1) \subseteq \mathcal{M}'_1. \quad (44)$$

Suppose that $\varphi(\mathbb{D}_{\leq 1}^{n \times n})$ is not an adjacent set. We show a contradiction as follows.

There exists a type one maximal set \mathcal{R}_1 containing 0 in $\mathbb{D}^{n \times n}$ with $\mathcal{R}_1 \neq \mathcal{M}_1$, such that $\varphi(\mathcal{R}_1) \not\subseteq \mathcal{M}'_1$. Otherwise, for any type one maximal set \mathcal{R}_1 containing 0 in $\mathbb{D}^{n \times n}$, we have $\varphi(\mathcal{R}_1) \subseteq \mathcal{M}'_1$. Since every matrix in $\mathbb{D}_{\leq 1}^{n \times n}$ is contained in some type one maximal set containing 0, we get $\varphi(\mathbb{D}_{\leq 1}^{n \times n}) \subseteq \mathcal{M}'_1$, a contradiction. By (43), it is easy to see that

$$\varphi(\mathcal{R}_1) \subseteq \mathcal{N}'.$$

Using Lemma 2.4(ii), there is $P_3 \in GL_n(\mathbb{D})$ such that $\mathcal{R}_1 = P_3 \mathcal{M}_2$ and $\mathcal{M}_1 = P_3 \mathcal{M}_1$. On the other hand, there is $Q_2 \in GL_{n'}(\mathbb{D}')$ such that $\mathcal{N}' = \mathcal{N}'_1 Q_2$. Modifying the map φ by the map $X \mapsto \varphi(P_3 X) Q_2^{-1}$. We get $\varphi(\mathcal{M}_2) \subseteq \mathcal{N}'_1$ and (44) holds. Therefore, we may assume that

$$\varphi(\mathcal{M}_2) \subseteq \mathcal{N}'_1, \quad \varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1 \quad \text{and} \quad \varphi(\mathcal{N}_1) \subseteq \mathcal{M}'_1. \quad (45)$$

By the conditions of this lemma, there exists $A_0 \in GL_n(\mathbb{D})$ such that $\text{rank}(\varphi(A_0)) = n$. Write $A_0 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = (\beta_1, \dots, \beta_n)$ where $\alpha_i \in \mathbb{D}^n$ and $\beta_i \in {}^n \mathbb{D}$; $\varphi(A_0) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where $A_{11} \in \mathbb{D}'^{1 \times 1}$ and $A_{22} \in \mathbb{D}'^{(m'-1) \times (n'-1)}$. Let $B = \begin{pmatrix} \alpha_1 \\ 0_{n-1, n} \end{pmatrix}$, $C = (\beta_1, 0, \dots, 0)$ and $J_n = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-2} & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then $B < A_0$ and $C < A_0$. By Corollary 4.7, we have $\varphi(B) < \varphi(A_0)$ and $\varphi(C) < \varphi(A_0)$. Since $\varphi(B) \in \mathcal{M}'_1$ and $\varphi(C) \in \mathcal{N}'_1$, it is clear that

$$\text{rank}(A_{21}, A_{22}) = \text{rank} \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = n - 1. \quad (46)$$

By (46) and $\text{rank}(\varphi(A_0)) = n$, we have either $A_{22} \cong \text{diag}(I_{n-1}, 0)$ or $A_{22} \cong \text{diag}(I_{n-2}, 0)$. By appropriate elementary row and column operations on matrices, we can obtain either $\text{diag}(I_n, 0)$ (if

$A_{22} \cong \text{diag}(I_{n-1}, 0)$ or $\text{diag}(J_n, 0)$ (if $A_{22} \cong \text{diag}(I_{n-2}, 0)$) from $\varphi(A_0)$. Since our matrix elementary operations do not change \mathcal{N}'_1 and \mathcal{M}'_1 , (45) still holds. Thus, without loss of generality we may assume either $\varphi(A_0) = \text{diag}(I_n, 0)$ or $\varphi(A_0) = \text{diag}(J_n, 0)$.

Case 1. $\varphi(A_0) = \text{diag}(I_n, 0)$. Since $\begin{pmatrix} 0 & \\ & \alpha_2 + \lambda\alpha_1 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix} < A_0$ for all $\lambda \in \mathbb{D}^*$, Corollary 4.7 implies that $\varphi\left(\begin{pmatrix} 0 & \\ & \alpha_2 + \lambda\alpha_1 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right) < \varphi(A_0)$, $\lambda \in \mathbb{D}^*$. Thus $\varphi(\mathcal{M}_2) \subseteq \mathcal{N}'_1$ implies that $\varphi\left(\begin{pmatrix} 0 & \\ & \alpha_2 + \lambda\alpha_1 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ Z_\lambda & 0_{m'-1,n'-1} \end{pmatrix}$, where $Z_\lambda \in {}^{m'-1}\mathbb{D}'$, $\lambda \in \mathbb{D}^*$. Since $\varphi\left(\begin{pmatrix} \lambda^{-1}\alpha_2 + \alpha_1 & \\ & 0 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right) < \varphi(A_0)$ for all $\lambda \in \mathbb{D}^*$, it follows from $\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1$ that $\varphi\left(\begin{pmatrix} \lambda^{-1}\alpha_2 + \alpha_1 & \\ & 0 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right) = \begin{pmatrix} 1 & T_\lambda \\ 0 & 0 \end{pmatrix}$, where $T_\lambda \in \mathbb{D}'^{n'-1}$, $\lambda \in \mathbb{D}^*$. By $|\mathbb{D}^*| \geq 3$ and the adjacency, there is $\lambda_0 \in \mathbb{D}^*$ such that $Z_{\lambda_0} \neq 0$ and $T_{\lambda_0} \neq 0$. By $\varphi\left(\begin{pmatrix} \lambda_0^{-1}\alpha_2 + \alpha_1 & \\ & 0 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right) \sim \varphi\left(\begin{pmatrix} 0 & \\ & \alpha_2 + \lambda_0\alpha_1 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right)$, we obtain $\begin{pmatrix} 1 & T_{\lambda_0} \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ Z_{\lambda_0} & 0_{m'-1,n'-2} \end{pmatrix}$, a contradiction.

Case 2. $\varphi(A_0) = \text{diag}(J_n, 0)$. By $\begin{pmatrix} \alpha_1 + \alpha_2 & \\ & 0 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix} < A_0$ and Corollary 4.7, $\varphi\left(\begin{pmatrix} \alpha_1 + \alpha_2 & \\ & 0 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right) < \varphi(A_0)$. It follows from $\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1$ that $\varphi\left(\begin{pmatrix} \alpha_1 + \alpha_2 & \\ & 0 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right) = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0_{m'-n,n'-n} \end{pmatrix}$ where $\alpha \in \mathbb{D}'^{n-1}$. Since $\begin{pmatrix} 0 & \\ & \alpha_1 + \alpha_2 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix} < A_0$ and Corollary 4.7, $\varphi\left(\begin{pmatrix} 0 & \\ & \alpha_1 + \alpha_2 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right) < \varphi(A_0)$. By $\varphi(\mathcal{M}_2) \subseteq \mathcal{N}'_1$, we get $\varphi\left(\begin{pmatrix} 0 & \\ & \alpha_1 + \alpha_2 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right) = \begin{pmatrix} \beta & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0_{m'-n,n'-n} \end{pmatrix}$ where $\beta \in {}^{n-1}\mathbb{D}'$. Since $\varphi\left(\begin{pmatrix} \alpha_1 + \alpha_2 & \\ & 0 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right) \sim \varphi\left(\begin{pmatrix} 0 & \\ & \alpha_1 + \alpha_2 \\ & & 0 \\ & & & 0_{n-2,n} \end{pmatrix}\right)$, $\begin{pmatrix} \alpha & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0_{m'-n,n'-n} \end{pmatrix} \sim \begin{pmatrix} \beta & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0_{m'-n,n'-n} \end{pmatrix}$, a contradiction.

Thus both Case 1 and Case 2 cannot occur. It follows that $\varphi(\mathbb{D}_{\leq 1}^{n \times n})$ is an adjacent set. \square

Lemma 4.12 *Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$, and let m', n', n be integers with $m', n' \geq n \geq 2$. Suppose that $\varphi : \mathbb{D}^{n \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism with $\varphi(0) = 0$, and there exists $A_0 \in \mathbb{D}^{n \times n}$ such that $\text{rank}(\varphi(A_0)) = n$. Assume further that φ satisfies the Condition (I). Then φ is non-degenerate.*

Proof. We prove that φ is non-degenerate by contradiction. Suppose that φ is degenerate. Then there exists a matrix $A \in \mathbb{D}_{\leq 1}^{n \times n}$ and there are two maximal sets \mathcal{R} and \mathcal{S} of different types in $\mathbb{D}'^{m' \times n'}$ such that

$$\varphi(\mathbb{B}_A) \subseteq \mathcal{R} \cup \mathcal{S} \text{ with } \varphi(A) \in \mathcal{R} \cap \mathcal{S}. \quad (47)$$

Without loss of generality, we assume that \mathcal{R} is of type one and \mathcal{S} is of type two. By $\text{rank}(\varphi(A_0)) = n$ and (1), we have $\text{rank}(A_0) = n$. Since $d(\varphi(A), 0) \leq 1$, we have either $d(\varphi(A_0), \varphi(A)) = n$ or $d(\varphi(A_0), \varphi(A)) = n - 1$. Note either $d(A_0, A) = n$ or $d(A_0, A) = n - 1$. Let $E_{ij} = E_{ij}^{n \times n}$ and $E'_{ij} = E_{ij}^{m' \times n'}$. We distinguish the following cases to show a contradiction.

Case 1. $d(A_0, A) = n$ and $d(\varphi(A_0), \varphi(A)) = n - 1$. Then $\text{rank}(\varphi(A)) = \text{rank}(A) = 1$ and $\varphi(A) < \varphi(A_0)$. By Lemma 4.4(c), there are invertible matrices P_1, Q_1 over \mathbb{D}' such that $\varphi(A) = P_1 E'_{11} Q_1$ and $\varphi(A_0) = P_1 \text{diag}(I_n, 0) Q_1$. Also, there are invertible matrices P_2, Q_2 over \mathbb{D} such that $P_2 A Q_2 = E_{11}$. Let $P_2 A_0 Q_2 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where $A_{11} \in \mathbb{D}^{1 \times 1}$ and $A_{22} \in \mathbb{D}^{(n-1) \times (n-1)}$. Put $J_n = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-2} & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

By $\text{rank}(A_0) = \text{rank}(A_0 - A) = n$, it is clear that $A_{22} \cong I_{n-1}$ or $A_{22} \cong \text{diag}(I_{n-2}, 0)$. Using appropriate elementary row and column operations on matrices, we can obtain either $\text{diag}(a, I_{n-1})$ where $a \notin \{0, 1\}$ (if $A_{22} \cong I_{n-1}$) or J_n (if $A_{22} \cong \text{diag}(I_{n-2}, 0)$) from $P_2 A_0 Q_2$. Moreover, $P_2 A Q_2 = E_{11}$ is unchanged. Thus, there are invertible matrices P_3, Q_3 over \mathbb{D} , such that $P_3 A Q_3 = E_{11}$ and either $P_3 A_0 Q_3 = \text{diag}(a, I_{n-1})$ or $P_3 A_0 Q_3 = J_n$. Replacing φ by the map $X \mapsto P_1^{-1} \varphi(P_3^{-1} X Q_3^{-1}) Q_1^{-1}$, we have that $A = E_{11}$, and either $A_0 = \text{diag}(a, I_{n-1})$ or $A_0 = J_n$. Moreover, $\varphi(E_{11}) = E'_{11}$ and $\varphi(\text{diag}(a, I_{n-1})) = \text{diag}(I_n, 0)$. On the other hand, (47) becomes $\varphi(\mathbb{D}_{\leq 1}^{n \times n} + E_{11}) \subseteq \mathcal{R} \cup \mathcal{S}$ with $\varphi(E_{11}) = E'_{11} \in \mathcal{R} \cap \mathcal{S}$.

Recall that φ satisfies the Condition (I). By $\varphi(E_{11}) = E'_{11}$, $\varphi(0) = 0$ and Corollary 2.2, we may assume with no loss of generality that

$$\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1, \quad \varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1. \quad (48)$$

Since $\mathcal{M}_1, \mathcal{N}_1 \subseteq \mathbb{D}_{\leq 1}^{n \times n} + E_{11}$, we get that $\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1 \cap (\mathcal{R} \cup \mathcal{S})$ and $\varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1 \cap (\mathcal{R} \cup \mathcal{S})$. Suppose that $\mathcal{M}'_1 \neq \mathcal{R}$. Then $\mathcal{M}'_1 \cap (\mathcal{R} \cup \mathcal{S}) = \mathcal{M}'_1 \cap \mathcal{S}$, thus $\ell := \mathcal{M}'_1 \cap \mathcal{S}$ is a line in $AG(\mathcal{M}'_1)$ and $\varphi(\mathcal{M}'_1) \subseteq \ell$, a contradiction to Lemma 4.8. Therefore, we must have $\mathcal{M}'_1 = \mathcal{R}$. Similarly, $\mathcal{N}'_1 = \mathcal{S}$. It follows that

$$\varphi(\mathbb{D}_{\leq 1}^{n \times n} + E_{11}) \subseteq \mathcal{M}'_1 \cup \mathcal{N}'_1. \quad (49)$$

Subcase 1.1. $A_0 = \text{diag}(a, I_{n-1})$. By (48), we have $\varphi(aE_{11}) = a^* E'_{11}$ where $a^* \in \mathbb{D}^*$. Since $aE_{11} < A_0$, Corollary 4.7 implies that $\varphi(aE_{11}) < \varphi(A_0) = \text{diag}(I_n, 0)$. Consequently, $a^* = 1$ and hence $\varphi(aE_{11}) = E'_{11}$, a contradiction to $\varphi(aE_{11}) \sim \varphi(E_{11})$.

Subcase 1.2. $A_0 = J_n$. By (49), we get $\varphi(E_{nn} + E_{11}) \in \mathcal{M}'_1 \cup \mathcal{N}'_1$. Without loss of generality, we assume that $\varphi(E_{nn} + E_{11}) \in \mathcal{M}'_1$. Since $d(E_{nn} + E_{11}, J_n) = n - 1$, $d(\varphi(E_{nn} + E_{11}), \text{diag}(I_n, 0)) \leq n - 1$. It follows that $\varphi(E_{nn} + E_{11}) = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}$, where $0 \neq \alpha \in \mathbb{D}'^{n'-1}$ because $\varphi(E_{nn} + E_{11}) \sim \varphi(E_{11}) = E'_{11}$. Since $d(E_{n1} + E_{11}, J_n) = n - 1$, $d(\varphi(E_{n1} + E_{11}), \text{diag}(I_n, 0)) \leq n - 1$. Thus $\varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1$ implies that $\varphi(E_{n1} + E_{11}) = \begin{pmatrix} 1 & 0 \\ \beta & 0 \end{pmatrix}$, where $0 \neq \beta \in {}^{m'-1}\mathbb{D}'$ because $\varphi(E_{n1} + E_{11}) \sim \varphi(E_{11})$. Since $\varphi(E_{nn} + E_{11}) \sim \varphi(E_{n1} + E_{11})$, one gets $\begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ \beta & 0 \end{pmatrix}$, a contradiction.

By Subcases 1.1-1.2, we always have a contradiction in Case 1.

Case 2. $d(A_0, A) = n$ and $d(\varphi(A_0), \varphi(A)) = n$. Let $\psi(X) = \varphi(X + A) - \varphi(A)$, $X \in \mathbb{D}^{n \times n}$. Then $\psi : \mathbb{D}^{n \times n} \rightarrow \mathbb{D}^{m' \times n'}$ is a graph homomorphism with $\psi(0) = 0$ and $\text{rank}(\psi(A_0 - A)) = n$. Moreover, from (47) we have

$$\psi(\mathbb{D}_{\leq 1}^{n \times n}) \subseteq \mathcal{R}' \cup \mathcal{S}' \quad \text{with} \quad 0 \in \mathcal{R}' \cap \mathcal{S}', \quad (50)$$

where $\mathcal{R}' = \mathcal{R} - \varphi(A)$ is a maximal set of type one, and $\mathcal{S}' = \mathcal{S} - \varphi(A)$ is a maximal set of type two.

Subcase 2.1. ψ satisfies the Condition (I). Then, by Lemma 4.10 we have

$$\dim(\psi(\mathcal{M}_1)) = \dim(\psi(\mathcal{N}_1)) = n. \quad (51)$$

By Lemma 1.1, without loss of generality, we may assume either $\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1$ with $\varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1$, or $\varphi(\mathcal{M}_1) \subseteq \mathcal{N}'_1$ with $\varphi(\mathcal{N}_1) \subseteq \mathcal{M}'_1$. We prove this result only for the first case; the second case is similar. From now on we assume that

$$\psi(\mathcal{M}_1) \subseteq \mathcal{M}'_1 \quad \text{and} \quad \psi(\mathcal{N}_1) \subseteq \mathcal{N}'_1. \quad (52)$$

By (50) and (52), one has $\psi(\mathcal{M}_1) \subseteq \mathcal{M}'_1 \cap (\mathcal{R}' \cup \mathcal{S}')$ and $\psi(\mathcal{N}_1) \subseteq \mathcal{N}'_1 \cap (\mathcal{R}' \cup \mathcal{S}')$. Clearly, $\mathcal{M}'_1 \cap \mathcal{R}' = \{0\} = \mathcal{N}'_1 \cap \mathcal{S}'$. We assert $\mathcal{M}'_1 = \mathcal{R}'$. Otherwise, we have that $\mathcal{M}'_1 \neq \mathcal{R}'$ and $\psi(\mathcal{M}_1) \subseteq$

$\mathcal{M}'_1 \cap (\mathcal{R}' \cup \mathcal{S}') = \mathcal{M}'_1 \cap \mathcal{S}'$. Hence Lemma 2.5 implies that $\ell := \mathcal{M}'_1 \cap \mathcal{S}'$ is a line in $AG(\mathcal{M}'_1)$, a contradiction to Lemma 4.8. Similarly, $\mathcal{N}'_1 = \mathcal{S}'$. Therefore,

$$\psi\left(\mathbb{D}_{\leq 1}^{n \times n}\right) \subseteq \mathcal{M}'_1 \cup \mathcal{N}'_1. \quad (53)$$

By ψ satisfying the Condition (I) and $\psi(\mathcal{M}_1) \subseteq \mathcal{M}'_1$, ψ maps any type two maximal set containing 0 into a type two maximal set. Thus (53) and Lemma 4.8 imply that ψ maps any type two maximal set containing 0 into \mathcal{N}'_1 . By (51), let $\{B_1, \dots, B_n\}$ be a maximal left linear independent subset in $\psi(\mathcal{M}_1)$, and let $\psi(C_i) = B_i$ where $C_i \in \mathcal{M}_1$, $i = 1, \dots, n$. Since $C_i \sim 0$ and ψ satisfies the Condition (I), there is a type two maximal set \mathcal{R}_i containing C_i and 0 in $\mathbb{D}^{n \times n}$, such that $\psi(\mathcal{R}_i) \subseteq \mathcal{N}'_1$, $i = 1, \dots, n$. Thus, $\{B_1, \dots, B_n\} \subset (\mathcal{M}'_1 \cap \mathcal{N}'_1) = \mathbb{D}'E'_{11}$, a contradiction.

Subcase 2.2. ψ does not satisfies the Condition (I). By Lemma 4.11, $\psi\left(\mathbb{D}_{\leq 1}^{n \times n}\right)$ is an adjacent set. Thus $\varphi(\mathbb{B}_A)$ is an adjacent set. Then, there is a maximal set \mathcal{M}' containing $\varphi(A)$ in $\mathbb{D}'^{m' \times n'}$, such that

$$\varphi(\mathbb{B}_A) \subseteq \mathcal{M}'. \quad (54)$$

Without loss of generality, we assume that \mathcal{M}' is of type one.

Since $\text{rank}(A) \leq 1$, there is a type one maximal set \mathcal{M}_A [resp. type two maximal set \mathcal{S}_A] containing A and 0 in $\mathbb{D}^{n \times n}$. Since φ satisfies the Condition (I), there are two maximal sets \mathcal{M}'_A and \mathcal{S}'_A of different types in $\mathbb{D}'^{m' \times n'}$, such that $\varphi(\mathcal{M}_A) \subseteq \mathcal{M}'_A$ and $\varphi(\mathcal{S}_A) \subseteq \mathcal{S}'_A$. Without loss of generality, we assume that \mathcal{M}'_A is of type one and \mathcal{S}'_A is of type two. Since $\mathcal{S}_A \subseteq \mathbb{D}_{\leq 1}^{n \times n} + A$, it follows from (54) that $\varphi(\mathcal{S}_A) \subseteq (\mathcal{S}'_A \cap \mathcal{M}')$. By Lemma 2.5, $\ell := \mathcal{S}'_A \cap \mathcal{M}'$ is a line in $AG(\mathcal{S}'_A)$, a contradiction to Lemma 4.8.

Therefore, there is always a contradiction in Case 2.

Case 3. $d(A_0, A) = n - 1$. Then $A < A_0$. Since $\text{rank}(\varphi(A_0)) = \text{rank}(A_0) = n$ and Corollary 4.7, we get $\varphi(A) < \varphi(A_0)$. By Lemma 4.4(c), there are invertible matrices P_1, Q_1 over \mathbb{D} such that $A = P_1 E_{11} Q_1$ and $A_0 = P_1 I_n Q_1$, and there are invertible matrices P_2, Q_2 over \mathbb{D}' such that $\varphi(A) = P_2 E'_{11} Q_2$ and $\varphi(A_0) = P_2 \text{diag}(I_n, 0) Q_2$. Replacing φ by the map $X \mapsto P_2^{-1} \varphi(P_1 X Q_1) Q_2^{-1}$, we have that $A = E_{11}$, $A_0 = I_n$, $\varphi(E_{11}) = E'_{11}$ and $\varphi(I_n) = \text{diag}(I_n, 0)$.

Since φ satisfies the Condition (I) and $\varphi(E_{11}) = E'_{11}$, Corollary 2.2 implies either $\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1$ with $\varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1$, or $\varphi(\mathcal{M}_1) \subseteq \mathcal{N}'_1$ with $\varphi(\mathcal{N}_1) \subseteq \mathcal{M}'_1$. We prove this result only for the first case; the second case is similar. From now on we assume that

$$\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1 \text{ and } \varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1.$$

Recalling (47), we have $\varphi\left(\mathbb{D}_{\leq 1}^{n \times n} + E_{11}\right) \subseteq \mathcal{R} \cup \mathcal{S}$ with $\varphi(E_{11}) = E'_{11} \in \mathcal{R} \cap \mathcal{S}$. Since $\mathcal{M}_1, \mathcal{N}_1 \subset (\mathbb{D}_{\leq 1}^{n \times n} + E_{11})$, we obtain that $\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1 \cap (\mathcal{R} \cup \mathcal{S}) = (\mathcal{M}'_1 \cap \mathcal{R}) \cup (\mathcal{M}'_1 \cap \mathcal{S})$ and $\varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1 \cap (\mathcal{R} \cup \mathcal{S}) = (\mathcal{N}'_1 \cap \mathcal{R}) \cup (\mathcal{N}'_1 \cap \mathcal{S})$.

We show $\mathcal{M}'_1 = \mathcal{R}$ by contradiction. Suppose that $\mathcal{M}'_1 \neq \mathcal{R}$. Then $\mathcal{M}'_1 \cap \mathcal{R} = \{E'_{11}\}$ and hence $\varphi(\mathcal{M}_1) \subseteq (\mathcal{M}'_1 \cap \mathcal{S})$. By Lemma 2.5, $\ell := \mathcal{M}'_1 \cap \mathcal{S}$ is a line in $AG(\mathcal{M}'_1)$, a contradiction to Lemma 4.8. Hence $\mathcal{M}'_1 = \mathcal{R}$. Similarly, $\mathcal{N}'_1 = \mathcal{S}$. Thus

$$\varphi\left(\mathbb{D}_{\leq 1}^{n \times n} + E_{11}\right) \subseteq \mathcal{M}'_1 \cup \mathcal{N}'_1.$$

Since $E_{11} + E_{22} \in (\mathbb{D}_{\leq 1}^{n \times n} + E_{11})$, we get $\varphi(E_{11} + E_{22}) \in \mathcal{M}'_1$ or $\varphi(E_{11} + E_{22}) \in \mathcal{N}'_1$. Without loss of generality, we assume that $\varphi(E_{11} + E_{22}) \in \mathcal{M}'_1$. Recall $\varphi(I_n) = \text{diag}(I_n, 0)$. We have $d(E_{11} + E_{22}, I_n) = n - 2$ and $d(\varphi(E_{11} + E_{22}), \text{diag}(I_n, 0)) \geq n - 1$, a contradiction.

Combining Cases 1-3, we always have a contradiction. Hence φ is non-degenerate. \square

Now, we prove Theorem 4.1 as follows.

Proof of Theorem 4.1. Suppose the graph homomorphism φ satisfies the Condition (I). Then, by Lemmas 4.10 and 4.12, φ is non-degenerate and $\dim(\varphi(\mathcal{M}_1)) = \dim(\varphi(\mathcal{N}_1)) = n$. By [14, Theorem 3.1], φ is of the form either (26) or (27). Now, we assume that the homomorphism φ does not satisfies the Condition (I). Then Lemma 4.11 implies that $\varphi(\mathbb{D}_{\leq 1}^{m \times n})$ is an adjacent set. \square

5 Degenerate graph homomorphisms

In this section, we discuss the degenerate graph homomorphisms. For a degenerate graph homomorphism $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$, there are no explicit algebraic formulas of φ . Thus, we discuss the ranges and some properties on degenerate graph homomorphisms.

Let $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ be a graph homomorphism. We called that φ maps distinct maximal sets of the same type [resp. different types] into distinct maximal sets of the same type [resp. different types], if for any two distinct maximal sets \mathcal{M} and \mathcal{N} of the same type [resp. different types] in $\mathbb{D}^{m \times n}$, there are two distinct maximal sets \mathcal{M}' and \mathcal{N}' of the same type [resp. different types] in $\mathbb{D}'^{m' \times n'}$, such that $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ and $\varphi(\mathcal{N}) \subseteq \mathcal{N}'$. We called that φ maps maximal sets of the same type [resp. different types] into maximal sets of the same type [resp. different types], if for any two distinct maximal sets \mathcal{M} and \mathcal{N} of the same type [resp. different types] in $\mathbb{D}^{m \times n}$, there are two maximal sets \mathcal{M}' and \mathcal{N}' of the same type [resp. different types] in $\mathbb{D}'^{m' \times n'}$, such that $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ and $\varphi(\mathcal{N}) \subseteq \mathcal{N}'$ (where \mathcal{M}' and \mathcal{N}' may be equal if \mathcal{M} and \mathcal{N} are of the same type).

Our main results in this section are the following two theorems.

Theorem 5.1 Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$, and let $m, n, m', n' \geq 2$ be integers with $m', n' \geq \min\{m, n\}$. Suppose $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a degenerate graph homomorphism with $\varphi(0) = 0$, and there exists $A_0 \in \mathbb{D}^{m \times n}$ such that $\text{rank}(\varphi(A_0)) = \min\{m, n\}$. Then $\varphi(\mathbb{D}_{\leq 1}^{m \times n})$ and $\varphi(\mathbb{B}_{A_0})$ are two adjacent sets.

Theorem 5.2 Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$, and let $m, n, m', n' \geq 2$ be integers with $\min\{m, n\} = 2$. Assume that $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a degenerate graph homomorphism. Then there are two fixed maximal sets \mathcal{M} and \mathcal{N} of different types containing 0 in $\mathbb{D}'^{m' \times n'}$, such that

$$\varphi(\mathbb{D}^{m \times n}) \subseteq (\mathcal{M} + R) \cup (\mathcal{N} + R), \quad (55)$$

where $R \in \mathbb{D}'^{m' \times n'}$ is fixed.

To prove Theorems 5.1 and 5.2, we need the following lemmas.

Lemma 5.3 Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$, and let $m, n, m', n' \geq 2$ be integers with $n, m', n' \geq m$. Suppose that $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism with $\varphi(0) = 0$, and there exists $A_0 \in \mathbb{D}^{m \times n}$ such that $\text{rank}(\varphi(A_0)) = m$. Assume further that $\varphi(\mathbb{D}_{\leq 1}^{m \times n})$ is not an adjacent set. Then:

- (i) φ maps distinct maximal sets of type one containing 0 into distinct maximal sets of the same type, and φ maps maximal sets of different types [resp. the same type] containing 0 into maximal sets

of different types [resp. the same type]. Moreover, if \mathcal{M} is a maximal set containing 0 in $\mathbb{D}^{m \times n}$ and $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ where \mathcal{M}' is a maximal set in $\mathbb{D}^{m' \times n'}$, then $\varphi(\mathcal{M})$ is not contained in any line in $AG(\mathcal{M}')$;

- (ii) if $A \in \mathbb{D}_1^{m \times n}$ and $d(A_0, A) = m - 1$, then $\varphi(\mathbb{B}_A)$ is not contained in a union of two maximal sets of different types containing $\varphi(A)$.

Proof. By Theorem 4.1 or Corollary 4.3, without loss of generality, we assume that $n > m$. There is $Q_1 \in GL_n(\mathbb{D})$ such that $A_0 = (A_1, 0)Q_1$ where $A_1 \in GL_m(\mathbb{D})$. Let $\psi(X) = \varphi(XQ_1)$ for all $X \in \mathbb{D}^{m \times n}$. Then ψ is a graph homomorphism with $\psi(0) = 0$, and $\text{rank}(\psi(A_1, 0)) = m = \text{rank}(A_1, 0)$. Since $\varphi(\mathbb{D}_{\leq 1}^{m \times n})$ is not an adjacent set, $\psi(\mathbb{D}_{\leq 1}^{m \times n})$ is not an adjacent set. Define the map $g : \mathbb{D}^{m \times m} \rightarrow \mathbb{D}^{m' \times n'}$ by

$$g(Y) = \psi((Y, 0)) = \varphi((Y, 0)Q_1), \quad Y \in \mathbb{D}^{m \times m}.$$

Then g is a graph homomorphism with $g(0) = 0$, $\text{rank}(g(A_1)) = m = \text{rank}(A_1)$. By Theorem 4.1 or Corollary 4.3, either $g(\mathbb{D}_{\leq 1}^{m \times m})$ is an adjacent set, or g is a distance preserving map.

We assert that $g(\mathbb{D}_{\leq 1}^{m \times m})$ is not an adjacent set. Otherwise, $g(\mathbb{D}_{\leq 1}^{m \times m}) \subseteq \mathcal{R}'$ where \mathcal{R}' is a maximal set containing 0 in $\mathbb{D}^{m' \times n'}$. We show a contradiction as follows.

By Lemma 1.1, $\mathcal{R}' = P_2\mathcal{M}'_1Q_2$ or $\mathcal{R}' = P_2\mathcal{N}'_1Q_2$ where $P_2 \in GL_{m'}(\mathbb{D}')$ and $Q_2 \in GL_{n'}(\mathbb{D}')$. Replacing the map g by the map $X \mapsto P_2^{-1}g(X)Q_2^{-1}$, we have either $\mathcal{R}' = \mathcal{M}'_1$ or $\mathcal{R}' = \mathcal{N}'_1$. Without loss of generality, we may assume that $\mathcal{R}' = \mathcal{M}'_1$. Then

$$g(\mathbb{D}_{\leq 1}^{m \times m}) \subseteq \mathcal{M}'_1. \quad (56)$$

Thus, we obtain

$$\psi(\mathcal{N}_j) \subseteq \mathcal{M}'_1, \quad j = 1, \dots, m. \quad (57)$$

Set $(\mathbb{D}_{\leq 1}^{m \times m}, 0) = \{(X, 0) : X \in \mathbb{D}_{\leq 1}^{m \times m}\} \subset \mathbb{D}_{\leq 1}^{m \times n}$. By (56), we have $\psi(\mathbb{D}_{\leq 1}^{m \times m}, 0) \subseteq \mathcal{M}'_1$. Let \mathcal{S} be any maximal set of type one containing 0 in $\mathbb{D}^{m \times n}$, and let $\psi(\mathcal{S}) \subseteq \mathcal{S}'$ where \mathcal{S}' is a maximal set in $\mathbb{D}^{m' \times n'}$. Then $|\mathcal{S} \cap \mathcal{N}_1| \geq 2$. Thus (57) implies that $|\mathcal{S}' \cap \mathcal{M}'_1| \geq 2$. Using Corollary 2.3, either $\mathcal{S}' = \mathcal{M}'_1$ or \mathcal{S}' is of type two. Put $(\mathcal{S}^{(m)}, 0) = \mathcal{S} \cap (\mathbb{D}_{\leq 1}^{m \times m}, 0)$. Then $\mathcal{S}^{(m)}$ is a maximal set of type one in $\mathbb{D}^{m \times m}$. From (56) we get $g(\mathcal{S}^{(m)}) \subseteq \mathcal{M}'_1$. Since $g(\mathcal{S}^{(m)}) \subseteq \mathcal{S}'$, $g(\mathcal{S}^{(m)}) \subseteq \mathcal{M}'_1 \cap \mathcal{S}'$. If \mathcal{S}' is of type two, then $\ell := \mathcal{M}'_1 \cap \mathcal{S}'$ is a line in $AG(\mathcal{M}'_1)$, which is a contradiction because Lemma 4.8. Thus, $\mathcal{S}' = \mathcal{M}'_1$. Then, ψ maps every maximal set of type one containing 0 in $\mathbb{D}^{m \times n}$ into \mathcal{M}'_1 . Since every matrix of rank one is contained in a maximal set of type one containing 0, we obtain $\psi(\mathbb{D}_{\leq 1}^{m \times n}) \subseteq \mathcal{M}'_1$. Since $\psi(\mathbb{D}_{\leq 1}^{m \times n})$ is not an adjacent set, we get a contradiction.

Therefore, g is a distance preserving map, and hence g is non-degenerate.

- (i). By Lemma 2.6(b), g maps two distinct maximal sets of different types [resp. the same type] containing 0 into two distinct maximal sets of different types [resp. the same type]. Since $g(Y) = \varphi((Y, 0)Q_1)$ ($Y \in \mathbb{D}^{m \times m}$), it is easy to see that φ maps two distinct maximal sets of type one containing 0 into two distinct maximal sets of the same type.

Let \mathcal{M} be any type one maximal set containing 0 in $\mathbb{D}^{m \times n}$, and let $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ where \mathcal{M}' is a maximal set containing 0 in $\mathbb{D}^{m' \times n'}$. Set $(\mathcal{M}^{(m)}, 0) = \mathcal{M} \cap (\mathbb{D}_{\leq 1}^{m \times m}, 0)$. Then $\mathcal{M}^{(m)}$ is a maximal set of type one containing 0 in $\mathbb{D}^{m \times m}$. Recall that $g(Y) = \varphi((Y, 0)Q_1)$ ($Y \in \mathbb{D}^{m \times m}$). We get $(\mathcal{M}^{(m)}, 0)Q_1 \subseteq \mathcal{M}Q_1 = \mathcal{M}$. Thus,

$$\varphi((\mathcal{M}^{(m)}, 0)Q_1) = g(\mathcal{M}^{(m)}) \subseteq \varphi(\mathcal{M}) \subseteq \mathcal{M}'.$$

By Lemma 2.6(c), $g(\mathcal{M}^{(m)})$ is not contained in any line in $AG(\mathcal{M}')$. It follows that $\varphi(\mathcal{M})$ is not contained in any line in $AG(\mathcal{M}')$.

Thus, if \mathcal{M} is a type one maximal set containing 0 in $\mathbb{D}^{m \times n}$ and $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ where \mathcal{M}' is a maximal set in $\mathbb{D}^{m' \times n'}$, then $\varphi(\mathcal{M})$ is not contained in any line in $AG(\mathcal{M}')$.

Next, we discuss maximal sets of type two. Let $\mathcal{S}_1, \mathcal{S}_2$ be any two distinct maximal sets of type two containing 0 in $\mathbb{D}^{m \times n}$, and let $\varphi(\mathcal{S}_i) \subseteq \mathcal{S}'_i$ where \mathcal{S}'_i is a maximal set in $\mathbb{D}^{m' \times n'}$, $i = 1, 2$. Assume that $\mathcal{R}_1, \mathcal{R}_2$ are any two distinct type one maximal sets containing 0 in $\mathbb{D}^{m \times n}$. By the above results, there are two distinct maximal sets $\mathcal{R}'_1, \mathcal{R}'_2$ containing 0 in $\mathbb{D}^{m' \times n'}$, such that \mathcal{R}'_1 and \mathcal{R}'_2 are of the same type and $\varphi(\mathcal{R}_i) \subseteq \mathcal{R}'_i$, $i = 1, 2$. By Corollary 2.3, we have $|\mathcal{S}'_i \cap \mathcal{R}'_j| \geq 2$, $i, j = 1, 2$. Thus $\mathcal{R}'_1 \neq \mathcal{R}'_2$ implies that \mathcal{S}'_i and \mathcal{R}'_i are of different types $i = 1, 2$. Consequently \mathcal{S}'_1 and \mathcal{S}'_2 are of the same type.

Therefore, we have proved that φ maps two distinct maximal sets of different types [resp. the same type] containing 0 into two maximal sets of different types [resp. the same type].

On the other hand, by Lemma 2.5, $\ell_1 := \mathcal{S}_1 \cap \mathcal{R}_1$ and $\ell_2 := \mathcal{S}_1 \cap \mathcal{R}_2$ are two distinct lines in $AG(\mathcal{S}_1)$. Also, $\ell'_1 := \mathcal{S}'_1 \cap \mathcal{R}'_1$ and $\ell'_2 := \mathcal{S}'_2 \cap \mathcal{R}'_2$ are two distinct lines in $AG(\mathcal{S}'_1)$. Since $\varphi(\ell_i) \subseteq \ell'_i$, $i = 1, 2$, it follows that $\varphi(\mathcal{S}_1)$ is not contained in any line in $AG(\mathcal{S}'_1)$ because two different lines have at most a common point. Thus, if \mathcal{S}_1 is a type two maximal set containing 0 in $\mathbb{D}^{m \times n}$ and $\varphi(\mathcal{S}_1) \subseteq \mathcal{S}'_1$ where \mathcal{S}'_1 is a maximal set in $\mathbb{D}^{m' \times n'}$, then $\varphi(\mathcal{S}_1)$ is not contained in any line in $AG(\mathcal{S}'_1)$.

Therefore, if \mathcal{M} is a maximal set containing 0 in $\mathbb{D}^{m \times n}$ and $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ where \mathcal{M}' is a maximal set in $\mathbb{D}^{m' \times n'}$, we have prove that $\varphi(\mathcal{M})$ is not contained in any line in $AG(\mathcal{M}')$. Thus, the (i) of this lemma is proved.

(ii). Suppose $A \in \mathbb{D}_1^{m \times n}$ and $d(A_0, A) = m - 1$. Then $A < A_0$. By Lemma 4.4(c), there are $P_3 \in GL_m(\mathbb{D})$ and $Q_3 \in GL_n(\mathbb{D})$ such that $A = P_3 E_{11} Q_3$ and $A_0 = P_3 (I_m, 0) Q_3$. Replacing φ by the map $X \mapsto \varphi(P_3 X Q_3)$, we can assume that $A_0 = (I_m, 0)$ and $A = E_{11}$.

Define the map $f : \mathbb{D}^{m \times m} \rightarrow \mathbb{D}^{m' \times n'}$ by $f(Y) = \varphi(Y, 0)$, $Y \in \mathbb{D}^{m \times m}$. Then f is a graph homomorphism such that $f(0) = 0$ and $\text{rank}(f(I_m)) = m$. By Theorem 4.1 or Corollary 4.3, either $f(\mathbb{D}_{\leq 1}^{m \times m})$ is an adjacent set or f is a distance preserving map. Similar to the proof on the homomorphism g , we can prove that f is a distance preserving map. Since every distance preserving map carries distinct maximal sets into distinct maximal sets, $f(\mathbb{D}_{\leq 1}^{m \times m} + E_{11}^{m \times m})$ is not contained in a union of two maximal sets of different types containing $f(E_{11}^{m \times m})$. Since $f(\mathbb{D}_{\leq 1}^{m \times m} + E_{11}^{m \times m}) = \varphi(\mathbb{D}_{\leq 1}^{m \times m} + E_{11}^{m \times m}, 0) \subseteq \varphi(\mathbb{B}_{E_{11}})$, $\varphi(\mathbb{B}_{E_{11}})$ is not contained in a union of two maximal sets of different types containing $\varphi(E_{11})$. Thus the (ii) of this lemma is proved. \square

By the symmetry of rows and columns of a matrix, we have similarly the following lemma.

Lemma 5.4 *Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$, and let $m, n, m', n' \geq 2$ be integers with $m, m', n' \geq n$. Suppose that $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}^{m' \times n'}$ is a graph homomorphism with $\varphi(0) = 0$, and there exists $A_0 \in \mathbb{D}^{m \times n}$ such that $\text{rank}(\varphi(A_0)) = n$. Assume further that $\varphi(\mathbb{D}_{\leq 1}^{m \times n})$ is not an adjacent set. Then:*

- (i) *φ maps distinct type two maximal sets containing 0 into distinct maximal sets of the same type, and φ maps maximal sets of different types [resp. the same type] containing 0 into maximal sets of different types [resp. the same type]. Moreover, if \mathcal{M} is a maximal set containing 0 in $\mathbb{D}^{m \times n}$ and $\varphi(\mathcal{M}) \subseteq \mathcal{M}'$ where \mathcal{M}' is a maximal set in $\mathbb{D}^{m' \times n'}$, then $\varphi(\mathcal{M})$ is not contained in any line in $AG(\mathcal{M}')$;*
- (ii) *if $A \in \mathbb{D}_1^{m \times n}$ and $d(A_0, A) = n - 1$, then $\varphi(\mathbb{B}_A)$ is not contained in a union of two maximal sets of*

different types containing $\varphi(A)$.

Now, we prove Theorem 5.1 as follows.

Proof of Theorem 5.1. We prove this theorem only for the case of $m = \min\{m, n\}$; the case of $n = \min\{m, n\}$ is similar by using Lemma 5.4. Since φ is degenerate, there exists a matrix $A \in \mathbb{D}_{\leq 1}^{m \times n}$ and there are two maximal sets \mathcal{M} and \mathcal{N} of different types in $\mathbb{D}'^{m' \times n'}$, such that

$$\varphi(\mathbb{B}_A) \subseteq \mathcal{M} \cup \mathcal{N} \text{ with } \varphi(A) \in \mathcal{M} \cap \mathcal{N}. \quad (58)$$

Without loss of generality, we assume that \mathcal{M} is of type one and \mathcal{N} is of type two. Since $\text{rank}(\varphi(A_0)) = m$ and $\text{rank}(\varphi(A)) \leq 1$, we have either $d(\varphi(A_0), \varphi(A)) = m$ or $m - 1$.

Step 1. In this step, we will prove that $\varphi(\mathbb{D}_{\leq 1}^{m \times n})$ is an adjacent set by contradiction. By Corollary 4.3, without loss of generality, we assume that $n > m$. Suppose that $\varphi(\mathbb{D}_{\leq 1}^{m \times n})$ is not an adjacent set. We distinguish the following two cases to show a contradiction.

Case 1.1. $d(\varphi(A_0), \varphi(A)) = m$. Let $\psi(X) = \varphi(X + A) - \varphi(A)$, $X \in \mathbb{D}^{m \times n}$. Then $\psi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism with $\psi(0) = 0$ and $\text{rank}(\psi(A_0 - A)) = m$. We show that $\psi(\mathbb{D}_{\leq 1}^{m \times n})$ is an adjacent set. Otherwise, if $\psi(\mathbb{D}_{\leq 1}^{m \times n})$ is not an adjacent set, then by Lemma 5.3(i), $\psi(\mathbb{D}_{\leq 1}^{m \times n})$ is not contained in a union of two maximal sets of different types containing 0, which implies that $\varphi(\mathbb{B}_A)$ is not contained in a union of two maximal sets of different types containing $\varphi(A)$, a contradiction to (58). Therefore, $\psi(\mathbb{D}_{\leq 1}^{m \times n})$ is an adjacent set. It follows that $\varphi(\mathbb{B}_A)$ is an adjacent set. Thus, there exists a maximal set \mathcal{S} such that

$$\varphi(\mathbb{B}_A) \subseteq \mathcal{S}.$$

Since $\varphi(\mathbb{D}_{\leq 1}^{m \times n})$ is not an adjacent set, A is of rank one. There is a type one maximal set \mathcal{M}_A [resp. type two maximal set \mathcal{S}_A] containing A and 0 in $\mathbb{D}^{m \times n}$. By Lemma 5.3(i), we have $\varphi(\mathcal{M}_A) \subseteq \mathcal{M}'_A$ and $\varphi(\mathcal{S}_A) \subseteq \mathcal{S}'_A$, where \mathcal{M}'_A and \mathcal{S}'_A are two maximal sets of different types in $\mathbb{D}'^{m' \times n'}$ containing $\varphi(A)$ and 0. Since $\mathcal{M}_A, \mathcal{S}_A \subseteq \mathbb{B}_A$, we get $\varphi(\mathcal{M}_A) \subseteq (\mathcal{M}'_A \cap \mathcal{S})$ and $\varphi(\mathcal{S}_A) \subseteq (\mathcal{S}'_A \cap \mathcal{S})$. Without loss of generality, we assume that \mathcal{M}'_A and \mathcal{S} are of the same type. Then Corollary 2.3 implies that $\mathcal{M}'_A \cap \mathcal{S} = \{\varphi(A)\}$ and $\varphi(\mathcal{M}_A) \subseteq \{\varphi(A)\}$, a contradiction.

Case 1.2. $d(\varphi(A_0), \varphi(A)) = m - 1$. Then, we have $\text{rank}(\varphi(A)) = \text{rank}(A) = 1$ and $\varphi(A) < \varphi(A_0)$. Write $E'_{ij} = E_{ij}^{m' \times n'}$ and $E_{ij} = E_{ij}^{m \times n}$. By Lemma 4.4(c), there are invertible matrices P_1, Q_1 over \mathbb{D}' such that

$$\varphi(A) = P_1 E'_{11} Q_1 \text{ and } \varphi(A_0) = P_1 \text{diag}(I_m, 0) Q_1.$$

Also, there are invertible matrices P_2, Q_2 over \mathbb{D} such that $P_2 A Q_2 = E_{11}$. Let $P_2 A_0 Q_2 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where $A_{11} \in \mathbb{D}^{1 \times 1}$ and $A_{22} \in \mathbb{D}^{(m-1) \times (n-1)}$. Clearly, $d(A_0, A) = m$ or $m - 1$. By Lemma 5.3(ii) and (58), we have $d(A_0, A) = m$. Thus, either $A_{22} \cong \text{diag}(I_{m-1}, 0)$ or $A_{22} \cong \text{diag}(I_{m-2}, 0)$. Put $J_m = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{m-2} & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

When $A_{22} \cong \text{diag}(I_{m-1}, 0)$, by appropriate elementary row and column operations on matrices, we can obtain either $(\text{diag}(a, I_{m-1}), 0)$ (where $a \in \mathbb{D}^*$ with $a \neq 1$) or $\begin{pmatrix} 0_1 & 0 & 1 & 0 \\ 0 & I_{m-1} & 0 & 0 \end{pmatrix}$ from $P_2 A_0 Q_2$. Moreover, $P_2 A Q_2 = E_{11}$ is unchanged.

When $A_{22} \cong \text{diag}(I_{m-2}, 0)$, by appropriate elementary row and column operations on matrices, we can get $(J_m, 0)$ from $P_2 A_0 Q_2$. Moreover, $P_2 A Q_2 = E_{11}$ is unchanged.

Thus, there are invertible matrices P_3, Q_3 over \mathbb{D} , such that $P_3 A Q_3 = E_{11}$ and either

$$P_3 A_0 Q_3 = (\text{diag}(a, I_{m-1}), 0), \text{ or } (J_m, 0), \text{ or } \begin{pmatrix} 0_1 & 0 & 1 & 0 \\ 0 & I_{m-1} & 0 & 0 \end{pmatrix}.$$

Replacing φ by the map $X \mapsto P_1^{-1} \varphi(P_3^{-1} X Q_3^{-1}) Q_1^{-1}$, we have $A = E_{11}$, and either $A_0 = (\text{diag}(a, I_{m-1}), 0)$, or $(J_m, 0)$, or $\begin{pmatrix} 0_1 & 0 & 1 & 0 \\ 0 & I_{m-1} & 0 & 0 \end{pmatrix}$. Moreover, we have

$$\varphi(E_{11}) = E'_{11} \text{ and } \varphi(A_0) = \text{diag}(I_m, 0). \quad (59)$$

On the other hand, (58) becomes

$$\varphi(\mathbb{B}_{E_{11}}) \subseteq \mathcal{M} \cup \mathcal{N} \text{ with } \varphi(E_{11}) \in \mathcal{M} \cap \mathcal{N}.$$

In $\mathbb{D}^{m \times n}$ [resp. $\mathbb{D}^{m' \times n'}$], by Corollary 2.2, there are only two maximal sets of different types containing E_{11} and 0 [resp. E'_{11} and 0], they are \mathcal{M}_1 and \mathcal{N}_1 [resp. \mathcal{M}'_1 and \mathcal{N}'_1]. Thus, by Lemma 5.3(i) and $\varphi(E_{11}) = E'_{11}$, we have either $\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1$ with $\varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1$, or $\varphi(\mathcal{M}_1) \subseteq \mathcal{N}'_1$ with $\varphi(\mathcal{N}_1) \subseteq \mathcal{M}'_1$. Without loss of generality, we assume that

$$\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1, \quad \varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1. \quad (60)$$

Since $\mathcal{M}_1, \mathcal{N}_1 \subseteq (\mathbb{D}_{\leq 1}^{m \times n} + E_{11})$, we get that $\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1 \cap (\mathcal{M} \cup \mathcal{N})$ and $\varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1 \cap (\mathcal{M} \cup \mathcal{N})$. Suppose that $\mathcal{M}'_1 \neq \mathcal{M}$. Then $\mathcal{M}'_1 \cap \mathcal{M} = \{E'_{11}\}$ and $\mathcal{M}'_1 \cap (\mathcal{M} \cup \mathcal{N}) = \mathcal{M}'_1 \cap \mathcal{N}$, thus $\ell := \mathcal{M}'_1 \cap \mathcal{N}$ is a line in $AG(\mathcal{M}'_1)$ and $\varphi(\mathcal{M}'_1) \subseteq \ell$. By Lemma 5.3(i), this is a contradiction. Therefore, we must have $\mathcal{M}'_1 = \mathcal{M}$. Similarly, $\mathcal{N}'_1 = \mathcal{N}$. It follows that

$$\varphi(\mathbb{B}_{E_{11}}) \subseteq \mathcal{M}'_1 \cup \mathcal{N}'_1. \quad (61)$$

In order to give a contradiction, we distinguish the following subcases.

Subcase 1.2.1. $A_0 = (\text{diag}(a, I_{m-1}), 0)$ where $a \in \mathbb{D}^*$ with $a \neq 1$. By (60), we have $\varphi(aE_{11}) = a^* E'_{11}$ where $a^* \in \mathbb{D}^*$. Since $aE_{11} < A_0$, Corollary 4.7 and (59) imply that $\varphi(aE_{11}) < \varphi(A_0) = \text{diag}(I_m, 0)$. Thus, $a^* = 1$ and hence $\varphi(aE_{11}) = E'_{11}$. Since $\varphi(aE_{11}) \sim \varphi(E_{11}) = E'_{11}$, we have a contradiction.

Subcase 1.2.2. $A_0 = (J_m, 0)$. By (61), we get $\varphi(E_{mm} + E_{11}) \in \mathcal{M}'_1 \cup \mathcal{N}'_1$. Without loss of generality, we assume that $\varphi(E_{mm} + E_{11}) \in \mathcal{M}'_1$. Since $d(E_{mm} + E_{11}, (J_m, 0)) = m - 1$, it follows from (59) that $d(\varphi(E_{mm} + E_{11}), \text{diag}(I_m, 0)) \leq m - 1$. Thus $\varphi(E_{mm} + E_{11}) = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}$ where $0 \neq \alpha \in \mathbb{D}^{n'-1}$ because $\varphi(E_{mm} + E_{11}) \sim \varphi(E_{11}) = E'_{11}$. Since $d(E_{m1} + E_{11}, (J_m, 0)) = m - 1$, from (59) we have $d(\varphi(E_{m1} + E_{11}), \text{diag}(I_m, 0)) \leq m - 1$. Thus $\varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1$ implies that $\varphi(E_{m1} + E_{11}) = \begin{pmatrix} 1 & 0 \\ \beta & 0 \end{pmatrix}$ where $0 \neq \beta \in {}^{m'-1}\mathbb{D}'$ because $\varphi(E_{m1} + E_{11}) \sim E'_{11}$. By $\varphi(E_{mm} + E_{11}) \sim \varphi(E_{m1} + E_{11})$, one gets $\begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ \beta & 0 \end{pmatrix}$, a contradiction.

Subcase 1.2.3. $A_0 = \begin{pmatrix} 0_1 & 0 & 1 & 0 \\ 0 & I_{m-1} & 0 & 0 \end{pmatrix}$. Let $B = E_{11} + E_{mm}$. Since $B \in \mathbb{B}_{E_{11}}$, from (61) we have $\varphi(B) \in \mathcal{M}'_1$ or $\varphi(B) \in \mathcal{N}'_1$.

Assume that $\varphi(B) \in \mathcal{M}'_1$. Since $d(B, A_0) = m - 1$, by (59) we get $d(\varphi(B), \text{diag}(I_m, 0)) \leq m - 1$. On the other hand, we have $\varphi(B) \sim \varphi(E_{11}) = E'_{11}$, which implies that $\varphi(B) = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}$ where $0 \neq \alpha \in \mathbb{D}^{n'-1}$. Let $C_1 = E_{11} + E_{m1}$. Since $C_1 \in \mathcal{N}_1$ and $\varphi(C_1) \sim \varphi(B)$, it follows from $\varphi(\mathcal{N}_1) \subseteq \mathcal{N}'_1$ that $\varphi(C_1) = c_1 E'_{11}$, where $c_1 \in \mathbb{D}^*$. Let \mathcal{M}_C be the type one maximal set containing C_1 and 0 in $\mathbb{D}^{m \times n}$. Then $\mathcal{M}_C \neq \mathcal{M}_1$.

Suppose $\varphi(\mathcal{M}_C) \subseteq \mathcal{M}'_C$ where \mathcal{M}'_C is a maximal set containing $\varphi(C_1)$ and 0 in $\mathbb{D}'^{m' \times n'}$. By Lemma 5.3(i) and (60), \mathcal{M}'_C is of type one and $\mathcal{M}'_C \neq \mathcal{M}'_1$. Thus, $\{0, \varphi(C_1)\} \subseteq \mathcal{M}'_C \cap \mathcal{M}'_1$, which is a contradiction to Corollary 2.3.

Assume that $\varphi(B) \in \mathcal{N}'_1$. Similarly, we have $\varphi(B) = \begin{pmatrix} 1 & 0 \\ \beta & 0 \end{pmatrix}$ where $0 \neq \beta \in {}^{m'-1}\mathbb{D}'$. Let $B_1 = E_{11} + E_{m1} + E_{mm} - E_{m,m+1}$. Since $B_1 \in \mathbb{B}_{E_{11}}$, by (61) we get $\varphi(B_1) \in \mathcal{M}'_1$ or $\varphi(B_1) \in \mathcal{N}'_1$. By $d(B_1, A_0) = m - 1$ and (59), we have $d(\varphi(B_1), \text{diag}(I_m, 0)) \leq m - 1$. Thus $\varphi(B_1) = \begin{pmatrix} 1 & \alpha_1 \\ 0 & 0 \end{pmatrix}$ or $\varphi(B_1) = \begin{pmatrix} 1 & 0 \\ \beta_1 & 0 \end{pmatrix}$, where $0 \neq \alpha_1 \in \mathbb{D}'^{m'-1}$ and $0 \neq \beta_1 \in {}^{m'-1}\mathbb{D}'$. Since $\varphi(B_1) \sim \varphi(B)$, we must have $\varphi(B_1) = \begin{pmatrix} 1 & 0 \\ \beta_1 & 0 \end{pmatrix}$. Let $B_2 = -E_{1m} + E_{1,m+1} \in \mathcal{M}_1$. Then $B_2 \sim B_1$ and $B_2 \sim E_{11}$, hence (59) and (60) imply that $\varphi(B_2) = b_2 E'_{11}$, where $b_2 \in \mathbb{D}'^*$ with $b_2 \neq 1$. However, we have $d(B_2, A_0) = m - 1$ and $d(\varphi(B_2), \varphi(A_0)) = d(b_2 E'_{11}, \text{diag}(I_m, 0)) = m$, a contradiction.

Combining Case 1.1 with Case 1.2, there is always a contradiction. Therefore, $\varphi(\mathbb{D}_{\leq 1}^{m \times n})$ must be an adjacent set.

Step 2. In this step, we will prove that $\varphi(\mathbb{B}_{A_0})$ is an adjacent set. By Step 1, $\varphi(\mathbb{D}_{\leq 1}^{m \times n})$ is an adjacent set. Thus, there is a maximal set \mathcal{M}' containing 0 in $\mathbb{D}'^{m' \times n'}$, such that $\varphi(\mathbb{D}_{\leq 1}^{m \times n}) \subseteq \mathcal{M}'$. There is an invertible matrix Q over \mathbb{D} such that $A_0 = (A_1, 0)Q$ where $A_1 \in GL_m(\mathbb{D})$.

Let $\varphi'(X) = \varphi(XQ)$ for all $X \in \mathbb{D}^{m \times n}$. Then φ' is a graph homomorphism, $\text{rank}(\varphi'(A_1, 0)) = m$ and

$$\varphi'(\mathbb{D}_{\leq 1}^{m \times n}) \subseteq \mathcal{M}'. \quad (62)$$

Let $f(Y) = \varphi'(Y + A_1, 0) - \varphi'(A_1, 0)$, $Y \in \mathbb{D}^{m \times m}$. Then f is a graph homomorphism such that $f(0) = 0$ and $\text{rank}(f(-A_1)) = m$. By (62), it is clear that f does not preserve distance 2. It follows from Corollary 4.3 that $f(\mathbb{D}_{\leq 1}^{m \times m})$ is an adjacent set. Hence there is a maximal set \mathcal{N}' containing 0 in $\mathbb{D}'^{m' \times n'}$, such that

$$f(\mathbb{D}_{\leq 1}^{m \times m}) \subseteq \mathcal{N}'.$$

Let $h(X) = \varphi'(X + (A_1, 0)) - \varphi'(A_1, 0)$, $X \in \mathbb{D}^{m \times n}$. Then h is a graph homomorphism with $h(0) = 0$. Moreover, $h(Y, 0) = f(Y)$ for all $Y \in \mathbb{D}^{m \times m}$. We assert $h(\mathbb{D}_{\leq 1}^{m \times n}) \subseteq \mathcal{N}'$. Otherwise, there exists a type one maximal set \mathcal{R} containing 0 in $\mathbb{D}^{m \times n}$, such that $h(\mathcal{R}) \subseteq \mathcal{S}$ where \mathcal{S} is a maximal set containing 0 in $\mathbb{D}'^{m' \times n'}$ and $\mathcal{S} \neq \mathcal{N}'$. Let $(\mathcal{R}^{(m)}, 0) = \mathcal{R} \cap (\mathbb{D}_{\leq 1}^{m \times m}, 0)$. Then $\mathcal{R}^{(m)}$ is a type one maximal set containing 0 in $\mathbb{D}^{m \times m}$, and hence $f(\mathcal{R}^{(m)}) \subseteq \mathcal{N}' \cap \mathcal{S}$. By Corollary 2.3, \mathcal{S} and \mathcal{N}' must be of different type. Thus $\ell := \mathcal{N}' \cap \mathcal{S}$ is a line in $AG(\mathcal{N}')$ by Lemma 2.5, a contradiction to Lemma 4.8. Therefore, we obtain $h(\mathbb{D}_{\leq 1}^{m \times n}) \subseteq \mathcal{N}'$, and hence $\varphi'(\mathbb{B}_{(A_1, 0)}) \subseteq \mathcal{N}' + \varphi'(A_1, 0)$. Consequently, $\varphi(\mathbb{B}_{A_0}) \subseteq \mathcal{N}' + \varphi(A_0)$ and hence $\varphi(\mathbb{B}_{A_0})$ is an adjacent set. \square

Next, we will prove Theorem 5.2. We need the following lemma.

Lemma 5.5 *Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$, and let $m, n, m', n' \geq 2$ be integers with $\min\{m, n\} = 2$. Suppose $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a degenerate graph homomorphism with $\varphi(0) = 0$, and there exists $A \in \mathbb{D}_2^{m \times n}$ such that $\text{rank}(\varphi(A)) = 2$. Then there are two fixed maximal sets \mathcal{M} and \mathcal{N} of different types in $\mathbb{D}'^{m' \times n'}$, such that $0 \in \mathcal{M} \cap \mathcal{N}$ and*

$$\varphi(\mathbb{D}^{m \times n}) \subseteq \mathcal{M} \cup (\mathcal{N} + \varphi(A)) = (\mathcal{M} + R) \cup (\mathcal{N} + R), \quad (63)$$

where $R \in \mathcal{M}$ is fixed and $\text{rank}(R) = 1$. Moreover, there is $\mathcal{M}' \in \{\mathcal{M}, \mathcal{N}\}$ such that $\varphi(\mathbb{D}_{\leq 1}^{m \times n}) \subseteq \mathcal{M}'$ and $\varphi(\mathbb{N}_A) \subseteq \mathcal{M}' \cap (\mathcal{N}' + R)$ where $\{\mathcal{M}', \mathcal{N}'\} = \{\mathcal{M}, \mathcal{N}\}$.

Proof. We prove this lemma only for the case $m = \min\{m, n\} = 2$; the case $n = \min\{m, n\} = 2$ is similar. From now on we assume that $m = \min\{m, n\} = 2$.

Step 1. By Theorem 5.1, $\varphi(\mathbb{D}_{\leq 1}^{2 \times n})$ and $\varphi(\mathbb{B}_A)$ are two adjacent sets. Then, there are maximal sets \mathcal{M} and \mathcal{N} containing 0 in $\mathbb{D}'^{m' \times n'}$, such that $\varphi(\mathbb{D}_{\leq 1}^{2 \times n}) \subseteq \mathcal{M}$ and $\varphi(\mathbb{B}_A) \subseteq \mathcal{N} + \varphi(A)$. By Lemma 1.1, $\mathcal{M} = P_1 \mathcal{M}'_1 Q_1$ or $\mathcal{M} = P_1 \mathcal{N}'_1 Q_1$ where $P_1 \in GL_{m'}(\mathbb{D}')$ and $Q_1 \in GL_{n'}(\mathbb{D}')$. Replacing φ by the map $X \mapsto P_1^{-1} \varphi(X) Q_1^{-1}$, we have either $\mathcal{M} = \mathcal{M}'_1$ or $\mathcal{M} = \mathcal{N}'_1$. We prove this lemma only for the case of $\mathcal{M} = \mathcal{M}'_1$; the case of $\mathcal{M} = \mathcal{N}'_1$ is similar. Now, we assume that $\mathcal{M} = \mathcal{M}'_1$.

Write $A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ where $\alpha_1, \alpha_2 \in \mathbb{D}^n$ are left linearly independent. Set $C_\lambda = \begin{pmatrix} -\alpha_1 + \lambda \alpha_2 \\ 0 \end{pmatrix}$ where $\lambda \in \mathbb{D}$. Since $C_\lambda + A \in \mathbb{B}_A \cap \mathbb{D}_{\leq 1}^{2 \times n}$ for all $\lambda \in \mathbb{D}$, $|(\mathcal{N} + \varphi(A)) \cap \mathcal{M}'_1| \geq 2$. Since $\text{rank}(\varphi(A)) = 2$, $\mathcal{N} + \varphi(A) \neq \mathcal{M}'_1$. Thus Corollary 2.3 implies that \mathcal{N} is of type two. By Lemma 1.1, $\mathcal{N} = \mathcal{N}'_1 Q_2$ where $Q_2 \in GL_{n'}(\mathbb{D}')$. Replacing φ by the map $X \mapsto \varphi(X) Q_2^{-1}$ ($X \in \mathbb{D}^{2 \times n}$), we may assume with no loss of generality that $\mathcal{N} = \mathcal{N}'_1$. Thus

$$\varphi(\mathbb{D}_{\leq 1}^{2 \times n}) \subseteq \mathcal{M}'_1, \quad \varphi(\mathbb{B}_A) \subseteq \mathcal{N}'_1 + \varphi(A). \quad (64)$$

We prove $\varphi(\mathbb{N}_A) \subseteq \mathcal{M}'_1$ as follows.

Write $\varphi(A) = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where $a_{11} \in \mathbb{D}'$. Since $C_\lambda + A \in \mathbb{N}_A \cap \mathbb{D}_{\leq 1}^{2 \times n}$ for all $\lambda \in \mathbb{D}$, $\varphi(C_\lambda + A) \in (\mathcal{N}'_1 + \varphi(A)) \cap \mathcal{M}'_1$ for all $\lambda \in \mathbb{D}$. Hence $\varphi(A)$ is of the form

$$\varphi(A) = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix}, \quad \text{where } a_{11} \in \mathbb{D}', A_{21} \neq 0 \text{ and } A_{12} \neq 0. \quad (65)$$

Otherwise, $A_{22} \neq 0$ implies that $(\mathcal{N}'_1 + \varphi(A)) \cap \mathcal{M}'_1 = \emptyset$, a contradiction.

Now, we affirm that $\text{rank}(\varphi(Z)) = 1$ for every $Z \in \mathbb{N}_A$. Suppose that $Z \in \mathbb{N}_A$ and $\text{rank}(\varphi(Z)) = 2$. Then $\text{rank}(Z) = 2$. We show a contradiction as follows. By Theorem 5.1, $\varphi(\mathbb{B}_Z)$ is an adjacent set. Thus, there exists a maximal set \mathcal{S} containing 0 in $\mathbb{D}'^{m' \times n'}$ such that

$$\varphi(\mathbb{B}_Z) \subseteq \mathcal{S} + \varphi(Z).$$

Write $Z = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$, where $\delta_1, \delta_2 \in \mathbb{D}^n$ are left linearly independent. Let $T_\lambda = \begin{pmatrix} -\delta_1 + \lambda \delta_2 \\ 0 \end{pmatrix}$ where $\lambda \in \mathbb{D}$. Since $T_\lambda + Z \in \mathbb{B}_Z \cap \mathbb{D}_{\leq 1}^{2 \times n}$ for all $\lambda \in \mathbb{D}$, $|(\mathcal{S} + \varphi(Z)) \cap \mathcal{M}'_1| \geq 2$. By $\text{rank}(\varphi(Z)) = 2$, we have $\mathcal{S} + \varphi(Z) \neq \mathcal{M}'_1$. Hence Corollary 2.3 implies that $\mathcal{S} + \varphi(Z)$ is a maximal set of type two. Applying Corollary 2.3 again, we get $|(\mathcal{S} + \varphi(Z)) \cap (\mathcal{N}'_1 + \varphi(A))| \leq 1$. Let $Z - A = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in \mathbb{D}_1^{2 \times n}$, $Z_1 = \begin{pmatrix} \gamma_1 \\ \gamma_1 - \gamma_2 \end{pmatrix}$, $Z_2 = \begin{pmatrix} \gamma_2 \\ 0 \end{pmatrix}$, $Y_1 = \begin{pmatrix} \gamma_1 \\ \gamma_1 \end{pmatrix}$ and $Y_2 = \begin{pmatrix} \gamma_2 \\ \gamma_2 \end{pmatrix}$. Then $Z_i + Z = Y_i + A$, $i = 1, 2$. Moreover, $Z_1 + Z \sim Z_2 + Z$. Clearly, $Z_i + Z \in \mathbb{B}_Z$ and $Y_i + A \in \mathbb{B}_A$, $i = 1, 2$. It follows that $|\mathbb{B}_Z \cap \mathbb{B}_A| \geq 2$ and hence $|(\mathcal{S} + \varphi(Z)) \cap (\mathcal{N}'_1 + \varphi(A))| \geq 2$, a contradiction. Therefore, $\text{rank}(\varphi(Z)) \neq 2$ for all $Z \in \mathbb{N}_A$. By $\text{rank}(\varphi(A)) = 2$ and $\varphi(A) \sim \varphi(Z)$, one gets $\text{rank}(\varphi(Z)) = 1$ for every $Z \in \mathbb{N}_A$.

Note that $\mathbb{N}_A \subset \mathbb{B}_A$. By (64) and (65), we obtain

$$\varphi(\mathbb{N}_A) \subseteq \left\{ \begin{pmatrix} y & A_{12} \\ 0 & 0 \end{pmatrix} : y \in \mathbb{D}' \right\} \subset \mathcal{M}'_1. \quad (66)$$

Step 2. Let $B \in \mathbb{D}_2^{2 \times n}$ with $B \neq A$ and $\text{rank}(\varphi(B)) = 2$. By Theorem 5.1, $\varphi(\mathbb{B}_B)$ is an adjacent set. Thus, there is a maximal set \mathcal{N}'' containing 0 in $\mathbb{D}'^{m' \times n'}$ such that

$$\varphi(\mathbb{B}_B) \subseteq \mathcal{N}'' + \varphi(B). \quad (67)$$

Write $B = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ where $\beta_1, \beta_2 \in \mathbb{D}^n$ are left linearly independent. Set $C_\lambda = \begin{pmatrix} -\beta_1 + \lambda\beta_2 \\ 0 \end{pmatrix}$ where $\lambda \in \mathbb{D}$. Since $C_\lambda + B \in \mathbb{B}_B \cap \mathbb{D}_{\leq 1}^{2 \times n}$ for all $\lambda \in \mathbb{D}$, $|(\mathcal{N}'' + \varphi(B)) \cap \mathcal{M}'_1| \geq 2$. Since $\text{rank}(\varphi(B)) = 2$, $\mathcal{N}'' + \varphi(B) \neq \mathcal{M}'_1$. It follows from Corollary 2.3 that \mathcal{N}'' is of type two. We prove $\mathcal{N}'' = \mathcal{N}'_1$ by contradiction as follows.

Suppose that $\mathcal{N}'' \neq \mathcal{N}'_1$. Write $\mathcal{N}'' = \{(yq_1, yq_2, \dots, yq_n) : y \in {}^m\mathbb{D}\}$ where $q_j \in \mathbb{D}'$ and $(q_2, \dots, q_n) \neq 0$. Then there is $Q_2 = \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix} \in GL_n(\mathbb{D}')$ such that $\mathcal{N}'' = \mathcal{N}'_2 Q_2$ and $\mathcal{N}'_1 = \mathcal{N}'_2 Q_2$. Modifying the map φ by the map $X \mapsto \varphi(X)Q_2^{-1}$ ($X \in \mathbb{D}^{2 \times n}$). We may assume with no loss of generality that $\mathcal{N}'' = \mathcal{N}'_2$ and (64)-(66) hold. Thus

$$\varphi(\mathbb{B}_B) \subseteq \mathcal{N}'_2 + \varphi(B).$$

Write $\varphi(B) = \begin{pmatrix} b_{11} & b_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{pmatrix}$ where $b_{11}, b_{12} \in \mathbb{D}'$. Since $C_\lambda + B \in \mathbb{B}_B \cap \mathbb{D}_{\leq 1}^{2 \times n}$ for all $\lambda \in \mathbb{D}$, $\varphi(C_\lambda + B) \in (\mathcal{N}'_2 + \varphi(B)) \cap \mathcal{M}'_1$ for all $\lambda \in \mathbb{D}$. Hence $\varphi(B)$ is of the form

$$\varphi(B) = \begin{pmatrix} b_{11} & b_{12} & B_{13} \\ 0 & B_{22} & 0 \end{pmatrix}, \quad (68)$$

where $B_{22} \neq 0$ and $(b_{11}, B_{13}) \neq (0, 0)$. Otherwise, we have $(B_{21}, B_{23}) \neq (0, 0)$, which implies that $(\mathcal{N}'_2 + \varphi(B)) \cap \mathcal{M}'_1 = \emptyset$, a contradiction. By $\varphi(B) \notin \mathcal{M}'_1$ and (66), we have $B \notin \mathbb{N}_A$ and hence $d(A, B) = 2$. Write $B - A = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where $v_1, v_2 \in \mathbb{D}^n$ are left linearly independent. Then $B \sim \begin{pmatrix} v_1 + \lambda v_2 \\ 0 \end{pmatrix} + A \sim A$ for all $\lambda \in \mathbb{D}$. Let $A_{12} = (a_{12}, A_{13})$ where $a_{12} \in \mathbb{D}'$. By (66), one has that

$$\varphi(B) = \begin{pmatrix} b_{11} & b_{12} & B_{13} \\ 0 & B_{22} & 0 \end{pmatrix} \sim \varphi\left(\begin{pmatrix} v_1 + \lambda v_2 \\ 0 \end{pmatrix} + A\right) =: \begin{pmatrix} \lambda^\sigma & a_{12} & A_{13} \\ 0 & 0 & 0 \end{pmatrix}, \text{ for all } \lambda \in \mathbb{D}, \quad (69)$$

where σ is an injection from \mathbb{D} to \mathbb{D}' . Hence

$$\text{rank}\begin{pmatrix} b_{11} - \lambda^\sigma & b_{12} - a_{12} & B_{13} - A_{13} \\ 0 & B_{22} & 0 \end{pmatrix} = 1, \text{ for all } \lambda \in \mathbb{D},$$

which is a contradiction because $B_{22} \neq 0$. Therefore, we must have $\mathcal{N}'' = \mathcal{N}'_1$.

Similar to the proof of (69), we have that $d(A, B) = 2$ and

$$\varphi(B) =: \begin{pmatrix} b_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{pmatrix} \sim \varphi\left(\begin{pmatrix} v_1 + \lambda v_2 \\ 0 \end{pmatrix} + A\right) =: \begin{pmatrix} \lambda^\mu & A_{12} \\ 0 & 0 \end{pmatrix} \text{ for all } \lambda \in \mathbb{D}, \quad (70)$$

where μ is an injection from \mathbb{D} to \mathbb{D}' . Therefore,

$$\text{rank}\begin{pmatrix} b_{11} - \lambda^\mu & B'_{12} - A_{12} \\ B'_{21} & B'_{22} \end{pmatrix} = 1, \text{ for all } \lambda \in \mathbb{D}.$$

Using Lemma 2.1 and $\text{rank}(\varphi(B)) = 2$, it is easy to see that $B'_{22} = 0$ and $B'_{12} = A_{12}$. Consequently, $\mathcal{N}'_1 + \varphi(B) = \mathcal{N}'_1 + \varphi(A)$, and (67) implies that

$$\varphi(\mathbb{N}_B) \subseteq \varphi(\mathbb{B}_B) \subseteq \mathcal{N}'_1 + \varphi(A), \text{ if } B \in \mathbb{D}_2^{2 \times n} \text{ with } B \neq A \text{ and } \text{rank}(\varphi(B)) = 2. \quad (71)$$

Step 3. Let $T \in \mathbb{D}_2^{2 \times n}$ with $\text{rank}(\varphi(T)) = 1$. We show $\varphi(T) \in \mathcal{M}'_1$ as follows.

Put $\varphi(T) = \begin{pmatrix} t_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ where $t_{11} \in \mathbb{D}'$. If $T \sim A$, then $T \in \mathbb{N}_A$ and hence $\varphi(T) \in \mathcal{M}'_1$ by (66). Now we assume $d(T, A) = 2$. Write $T - A = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$ where $\delta_1, \delta_2 \in \mathbb{D}^n$ are left linearly independent. Then

$T \sim \begin{pmatrix} \delta_1 + \lambda\delta_2 & \\ & 0 \end{pmatrix} + A \sim A$ for all $\lambda \in \mathbb{D}$. By (66), we can let $\varphi\left(\begin{pmatrix} \delta_1 + \lambda\delta_2 & \\ & 0 \end{pmatrix} + A\right) = \begin{pmatrix} \lambda^\tau & A_{12} \\ 0 & 0 \end{pmatrix}$ for all $\lambda \in \mathbb{D}$, where τ is an injection from \mathbb{D} to \mathbb{D}' . Since $\varphi(T) \sim \begin{pmatrix} \lambda^\tau & A_{12} \\ 0 & 0 \end{pmatrix}$ for all $\lambda \in \mathbb{D}$, we get that

$$\text{rank} \begin{pmatrix} t_{11} - \lambda^\tau & T_{12} - A_{12} \\ T_{21} & T_{22} \end{pmatrix} = \text{rank} \begin{pmatrix} t_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = 1, \text{ for all } \lambda \in \mathbb{D}.$$

By Lemma 2.1 and $A_{12} \neq 0$, we obtain $(T_{21}, T_{22}) = 0$. Then $\varphi(T) = \begin{pmatrix} t_{11} & T_{12} \\ 0 & 0 \end{pmatrix} \in \mathcal{M}'_1$. Hence

$$\varphi(T) \in \mathcal{M}'_1, \text{ if } T \in \mathbb{D}_2^{2 \times n} \text{ and } \text{rank}(\varphi(T)) = 1. \quad (72)$$

By (64), (71) and (72), we have that $\varphi(X) \in \mathcal{N}'_1 + \varphi(A)$ whenever $\text{rank}(\varphi(X)) = 2$, and $\varphi(X) \in \mathcal{M}'_1$ whenever $\text{rank}(\varphi(X)) \neq 2$. Thus

$$\varphi(\mathbb{D}^{2 \times n}) \subseteq \mathcal{M}'_1 \cup (\mathcal{N}'_1 + \varphi(A)).$$

Recall that $\varphi(A) = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix}$ where $A_{21} \neq 0$ and $A_{12} \neq 0$. Let $R = \begin{pmatrix} a_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \in \mathcal{M}'_1$. Then $\text{rank}(R) = 1$, $\mathcal{M}'_1 = \mathcal{M}'_1 + R$ and $\mathcal{N}'_1 + \varphi(A) = \mathcal{N}'_1 + R$, and hence $\mathcal{M}'_1 \cup (\mathcal{N}'_1 + \varphi(A)) = (\mathcal{M}'_1 + R) \cup (\mathcal{N}'_1 + R)$. Therefore, (63) holds. By (64) and (66), we have $\varphi(\mathbb{D}_{\leq 1}^{2 \times n}) \subseteq \mathcal{M}'_1$ and $\varphi(\mathbb{N}_A) \subseteq \mathcal{M}'_1 \cap (\mathcal{N}'_1 + R)$. Thus, there is $\mathcal{M}' \in \{\mathcal{M}, \mathcal{N}\}$ such that $\varphi(\mathbb{D}_{\leq 1}^{2 \times n}) \subseteq \mathcal{M}'$ and $\varphi(\mathbb{N}_A) \subseteq \mathcal{M}' \cap (\mathcal{N}' + R)$ where $\{\mathcal{M}', \mathcal{N}'\} = \{\mathcal{M}, \mathcal{N}\}$. \square

Now, we prove Theorem 5.2 as follows.

Proof of Theorem 5.2. Note that the map $\varphi(X) - \varphi(0)$ is also a degenerate graph homomorphism. By the map $X \mapsto \varphi(X) - \varphi(0)$, we may assume with no loss of generality that $\varphi(0) = 0$. We prove this theorem only for the case of $m = 2$; the case of $n = 2$ is similar. From now on we assume that $m = \min\{m, n\} = 2$. Clearly, $1 \leq \text{diam}(\varphi(\mathbb{D}^{2 \times n})) \leq 2$.

Case 1. There exists $A \in \mathbb{D}_2^{2 \times n}$ such that $\text{rank}(\varphi(A)) = 2$. By Lemma 5.5, this theorem holds.

Case 2. There is no $A \in \mathbb{D}_2^{2 \times n}$ such that $\text{rank}(\varphi(A)) = 2$. Then

$$\text{rank}(\varphi(X)) \leq 1, \text{ for all } X \in \mathbb{D}^{2 \times n}. \quad (73)$$

If $\text{diam}(\varphi(\mathbb{D}^{2 \times n})) = 1$, then $\varphi(\mathbb{D}^{2 \times n})$ is an adjacent set, and hence this theorem holds. From now on we assume that $\text{diam}(\varphi(\mathbb{D}^{2 \times n})) = 2$. Then, there are two matrices $B_1, B_2 \in \mathbb{D}^{2 \times n}$ such that $d(\varphi(B_1), \varphi(B_2)) = 2 = d(B_1, B_2)$. Let $\psi(X) = \varphi(X + B_1) - \varphi(B_1)$, $X \in \mathbb{D}^{2 \times n}$. Then $\psi : \mathbb{D}^{2 \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism with $\psi(0) = 0$ and $\text{rank}(\psi(B_2 - B_1)) = 2$.

Subcase 2.1. ψ is degenerate. By Lemma 5.5, there are two fixed maximal sets \mathcal{M} and \mathcal{N} of different types containing 0, such that $\psi(\mathbb{D}^{2 \times n}) \subseteq (\mathcal{M} + R_1) \cup (\mathcal{N} + R_1)$, where $R_1 \in \mathbb{D}'^{m' \times n'}$ is fixed and $\text{rank}(R_1) = 1$. Hence $\varphi(\mathbb{D}^{2 \times n}) \subseteq (\mathcal{M} + R_1 + \varphi(A)) \cup (\mathcal{N} + R_1 + \varphi(B_1))$ and this theorem holds.

Subcase 2.2. ψ is non-degenerate. By Lemma 3.3, there exist invertible matrices T_1, T_2 over \mathbb{D}' and invertible matrices T_3, T_4 over \mathbb{D} , such that either

$$\psi(T_3^{-1} \mathcal{M}_i T_4^{-1}) \subseteq T_1 \mathcal{M}'_i T_2 \text{ with } \psi(T_3^{-1} \mathcal{N}_i T_4^{-1}) \subseteq T_1 \mathcal{N}'_i T_2 \text{ (} i = 1, 2\text{)}, \quad (74)$$

or

$$\psi(T_3^{-1} \mathcal{M}_i T_4^{-1}) \subseteq T_1 \mathcal{N}'_i T_2 \text{ with } \psi(T_3^{-1} \mathcal{N}_i T_4^{-1}) \subseteq T_1 \mathcal{M}'_i T_2 \text{ (} i = 1, 2\text{)}. \quad (75)$$

Without loss of generality, we may assume that (74) holds and all T_1, T_2, T_3, T_4 are identity matrices. Then $\psi(\mathcal{M}_i) \subseteq \mathcal{M}'_i$ with $\psi(\mathcal{N}_i) \subseteq \mathcal{N}'_i$, $i = 1, 2$. Since $\mathcal{M}_i \cap \mathcal{N}_j = \mathbb{D}E_{ij}$ and $\mathcal{M}'_i \cap \mathcal{N}'_j = \mathbb{D}'E'_{ij}$, $1 \leq i, j \leq 2$, we can let $\psi(xE_{ij}) = x^{\sigma_{ij}}E'_{ij}$, $x \in \mathbb{D}$, where $\sigma_{ij} : \mathbb{D} \rightarrow \mathbb{D}'$ is an injective map with $0^{\sigma_{ij}} = 0$. By the definition of ψ , we obtain that

$$\varphi(xE_{ij} + A) = x^{\sigma_{ij}}E'_{ij} + \varphi(B_1), \quad x \in \mathbb{D}, i, j = 1, 2.$$

Using (73), we have $\text{rank}(x^{\sigma_{ij}}E'_{ij} + \varphi(B_1)) \leq 1$, $x \in \mathbb{D}$, $i, j = 1, 2$. Applying Lemma 2.1, it is easy to verify that $\varphi(B_1) = 0$. Thus, $\text{rank}(\varphi(B_2)) = d(\varphi(B_2), \varphi(B_1)) = 2$, a contradiction to (73). Therefore, Subcase 2.2 does not happen. \square

By Theorem 5.1, the following corollary is obvious.

Corollary 5.6 *Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$, and let $m, n, m', n' \geq 2$ be integers with $m', n' \geq \min\{m, n\}$. Suppose $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism with $\varphi(0) = 0$, and there exists $A_0 \in \mathbb{D}^{m \times n}$ such that $\text{rank}(\varphi(A_0)) = \min\{m, n\}$. If $\varphi(\mathbb{D}_{\leq 1}^{m \times n})$ or $\varphi(\mathbb{B}_{A_0})$ is not an adjacent set, then φ is non-degenerate.*

The following result is a generalization of [17, Theorem 1.1] (which is due to Huang and Šemrl).

Corollary 5.7 *Let \mathbb{D}, \mathbb{D}' be division rings with $|\mathbb{D}| \geq 4$, and let $m', n' \geq 2$. Suppose $\varphi : \mathbb{D}^{2 \times 2} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism. Then either φ is a distance preserving map (which is of the form either (28) or (29)), or there are two fixed maximal sets \mathcal{M} and \mathcal{N} of different types in $\mathbb{D}'^{m' \times n'}$, such that $0 \in \mathcal{M} \cap \mathcal{N}$ and*

$$\varphi(\mathbb{D}^{2 \times 2}) \subseteq (\mathcal{M} + R) \cup (\mathcal{N} + R), \quad (76)$$

where $R \in \mathbb{D}'^{m' \times n'}$ is fixed.

Proof. Assume that $\text{diam}(\varphi(\mathbb{D}^{2 \times 2})) = 1$. Then $\varphi(\mathbb{D}^{2 \times 2})$ is an adjacent set, and hence (76) holds. From now on we assume that $\text{diam}(\varphi(\mathbb{D}^{2 \times 2})) = 2$. Then there are two matrices $A_0, B_0 \in \mathbb{D}^{2 \times 2}$ such that $d(\varphi(A_0), \varphi(B_0)) = \text{rank}(\varphi(B_0) - \varphi(A_0)) = 2$. Let $\psi(X) = \varphi(X + A_0) - \varphi(A_0)$, $X \in \mathbb{D}^{2 \times 2}$. Then $\psi : \mathbb{D}^{2 \times 2} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism with $\psi(0) = 0$ and $\text{rank}(\psi(B_0 - A_0)) = 2$. By Corollary 4.3, either ψ is a distance preserving map or $\psi(\mathbb{D}_{\leq 1}^{2 \times 2})$ is an adjacent set.

Assume that ψ is a distance preserving map. Then φ is also a distance preserving map. By Corollary 4.2, φ is of the form either (28) or (29). Now, we assume that $\psi(\mathbb{D}_{\leq 1}^{2 \times 2})$ is an adjacent set. By Theorem 5.2 or Lemma 5.5, there are two fixed maximal sets \mathcal{M} and \mathcal{N} of different types in $\mathbb{D}'^{m' \times n'}$, such that $0 \in \mathcal{M} \cap \mathcal{N}$ and (76) holds. \square

Finally, we discuss the case of finite fields. For the case of finite fields, we have the following better results.

Theorem 5.8 *Let \mathbb{D}, \mathbb{D}' be two finite fields with $|\mathbb{D}| > |\mathbb{D}'|$, and let $m, n, m', n' \geq 2$ be integers. Then every graph homomorphism from $\mathbb{D}^{m \times n}$ to $\mathbb{D}'^{m' \times n'}$ is a (vertex) colouring.*

Proof. Suppose $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a graph homomorphism. Without loss of generality, we may assume that $\varphi(0) = 0$ and $\varphi(\mathcal{M}_1) \subseteq \mathcal{M}'_1$. If ℓ is a line in $AG(\mathcal{M}_1)$, then $|\mathbb{D}| > |\mathbb{D}'|$ implies that $\varphi(\ell)$ is not contained in any line in $AG(\mathcal{M}'_1)$, and hence $\varphi(\ell)$ contains at least three noncollinear points in $AG(\mathcal{M}'_1)$.

Let $A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \in \mathbb{D}_1^{m \times n}$. Without loss of generality, we assume $\alpha_m \neq 0$. Then $\alpha_i = k_i \alpha_m$, $i = 1, \dots, m-1$. We have $A \sim \begin{pmatrix} \lambda \alpha_m \\ 0 \end{pmatrix}$ for all $\lambda \in \mathbb{D}$. Since $\ell_1 := \left\{ \begin{pmatrix} \lambda \alpha_m \\ 0 \end{pmatrix} : \lambda \in \mathbb{D} \right\}$ is a line in $AG(\mathcal{M}_1)$, it follows from above result that $\varphi(A)$ is adjacent with three noncollinear points in $AG(\mathcal{M}'_1)$. By Lemma 2.8, $\varphi(A) \in \mathcal{M}'_1$ for all $A \in \mathbb{D}_1^{m \times n}$. Therefore, we obtain $\varphi(\mathbb{D}_{\leq 1}^{m \times n}) \subseteq \mathcal{M}'_1$.

Suppose that $\varphi(\mathbb{D}_{\leq k-1}^{m \times n}) \subseteq \mathcal{M}'_1$ where $2 \leq k \leq m$. Let $B \in \mathbb{D}_k^{m \times n}$. Note that $P(\mathbb{D}_{\leq k-1}^{m \times n})Q = \mathbb{D}_{\leq k-1}^{m \times n}$ for any $P \in GL_m(\mathbb{D})$ and $Q \in GL_n(\mathbb{D})$. Without loss of generality, we can assume that $B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$ where $B_1 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} \in \mathbb{D}_k^{k \times n}$. Put $C_1 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{k-1} \end{pmatrix}$. Then $B \sim \begin{pmatrix} C_1 + \begin{pmatrix} \lambda \alpha_k \\ 0 \end{pmatrix} \end{pmatrix}$ for all $\lambda \in \mathbb{D}$. Let $\ell_2 = \left\{ \begin{pmatrix} C_1 + \begin{pmatrix} \lambda \alpha_k \\ 0 \end{pmatrix} \end{pmatrix} : \lambda \in \mathbb{D} \right\} \subset \mathbb{D}_{\leq k-1}^{m \times n}$. Then ℓ_2 is a line in some affine geometry on a maximal set in $\mathbb{D}^{m \times n}$. By the induction hypothesis, we have $\varphi(\ell_2) \subseteq \mathcal{M}'_1$. Since $|\mathbb{D}| > |\mathbb{D}'|$, $\varphi(\ell_2)$ contains at least three noncollinear points in $AG(\mathcal{M}'_1)$. By Lemma 2.8 and $\varphi(B) \sim Y$ for any $Y \in \varphi(\ell_2)$, we obtain $\varphi(B) \in \mathcal{M}'_1$ for any $B \in \mathbb{D}_k^{m \times n}$. Hence $\varphi(\mathbb{D}_{\leq k}^{m \times n}) \subseteq \mathcal{M}'_1$. Taking $k = m$, we get $\varphi(\mathbb{D}^{m \times n}) \subseteq \mathcal{M}'_1$. \square

Let $\lceil x \rceil$ denote the smallest integer at least as large as x .

Theorem 5.9 *Let \mathbb{D}, \mathbb{D}' be two finite fields with $4 \leq |\mathbb{D}| \leq |\mathbb{D}'| \leq (|\mathbb{D}| - 1)\lceil (|\mathbb{D}| + 1)/2 \rceil$, and let $m, n, m', n' \geq 2$ be integers. Suppose $\varphi : \mathbb{D}^{m \times n} \rightarrow \mathbb{D}'^{m' \times n'}$ is a degenerate graph homomorphism. Then there are two fixed maximal sets \mathcal{M} and \mathcal{N} of different types containing 0 in $\mathbb{D}'^{m' \times n'}$, such that*

$$\varphi(\mathbb{D}^{m \times n}) \subseteq (\mathcal{M} + R) \cup (\mathcal{N} + R), \quad (77)$$

where $R \in \mathbb{D}'^{m' \times n'}$ is fixed.

Proof. When $\min\{m, n\} = 2$, this theorem holds by Theorem 5.2. From now on we assume that $m, n > 2$. By Theorem 5.2, there are two fixed maximal sets \mathcal{M} and \mathcal{N} of different types containing 0 in $\mathbb{D}'^{m' \times n'}$, such that $\varphi\left(\begin{pmatrix} \mathbb{D}^{2 \times n} \\ 0 \end{pmatrix}\right) \subseteq (\mathcal{M} + R) \cup (\mathcal{N} + R)$, where $R \in \mathbb{D}'^{m' \times n'}$ is fixed. Suppose that $3 \leq k \leq m$ and

$$\varphi\left(\begin{pmatrix} \mathbb{D}^{(k-1) \times n} \\ 0 \end{pmatrix}\right) \subseteq (\mathcal{M} + R) \cup (\mathcal{N} + R). \quad (78)$$

We prove $\varphi\left(\begin{pmatrix} \mathbb{D}^{k \times n} \\ 0 \end{pmatrix}\right) \subseteq (\mathcal{M} + R) \cup (\mathcal{N} + R)$ as follows.

$$\begin{aligned} \text{Let } A = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \in \mathbb{D}^{m \times n}, \text{ where } A_1 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} \in \mathbb{D}^{k \times n} \text{ with } \alpha_k \neq 0. \text{ Put } B_1 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{k-1} \end{pmatrix}. \text{ Then} \\ A \sim \begin{pmatrix} B_1 + \lambda \begin{pmatrix} \alpha_k \\ 0 \\ 0 \end{pmatrix} \\ 0 \end{pmatrix}, \quad A \sim \begin{pmatrix} B_1 + \lambda \begin{pmatrix} 0_{1,n} \\ \alpha_k \\ 0 \end{pmatrix} \\ 0 \end{pmatrix}, \quad A \sim \begin{pmatrix} B_1 + \lambda \begin{pmatrix} d\alpha_k \\ \alpha_k \\ 0 \end{pmatrix} \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{D}, d \in \mathbb{D}^*. \end{aligned} \quad (79)$$

Let

$$\ell_1 = \left\{ \begin{pmatrix} B_1 + \lambda \begin{pmatrix} \alpha_k \\ 0 \\ 0 \end{pmatrix} \\ 0 \end{pmatrix} : \lambda \in \mathbb{D} \right\}, \quad \ell_2 = \left\{ \begin{pmatrix} B_1 + \lambda \begin{pmatrix} 0_{1,n} \\ \alpha_k \\ 0 \end{pmatrix} \\ 0 \end{pmatrix} : \lambda \in \mathbb{D} \right\},$$

$$\ell'_d = \left\{ \left(B_1 + \lambda \begin{pmatrix} d\alpha_k \\ \alpha_k \\ 0 \end{pmatrix} : \lambda \in \mathbb{D} \right), d \in \mathbb{D}^* \right\}.$$

Then ℓ_1, ℓ_2, ℓ'_d ($d \in \mathbb{D}^*$) are $|\mathbb{D}| + 1$ distinct lines in affine geometries on $|\mathbb{D}| + 1$ maximal sets in $\mathbb{D}^{m \times n}$. Moreover, $\ell_1 \cap \ell_2 = \ell_i \cap \ell'_d = \ell'_{d_1} \cap \ell'_{d_2} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$ for $i = 1, 2$ and $d, d_1, d_2 \in \mathbb{D}^*$ with $d_1 \neq d_2$. Clearly, $\varphi(\ell_i)$ or ℓ'_d is contained in a maximal set in $\mathbb{D}'^{m' \times n'}$, $i = 1, 2, d \in \mathbb{D}^*$. Since $\ell_i, \ell'_d \subseteq \begin{pmatrix} \mathbb{D}^{(k-1) \times n} \\ 0 \end{pmatrix}$, it follows from (78) that $\varphi(\ell_i) \subseteq \mathcal{M} + R$ or $\varphi(\ell_i) \subseteq \mathcal{N} + R$, $i = 1, 2$; $\varphi(\ell'_d) \subseteq \mathcal{M} + R$ or $\varphi(\ell'_d) \subseteq \mathcal{N} + R$, $d \in \mathbb{D}^*$.

Let either $\mathcal{M}' = \mathcal{M} + R$ or $\mathcal{M}' = \mathcal{N} + R$. Suppose that there is some k or d such that $\varphi(\ell_k)$ or ℓ'_d is not contained in a line in $AG(\mathcal{M}')$. Then, from (79) $\varphi(A)$ is adjacent with three noncollinear points in $AG(\mathcal{M}')$. By Lemma 2.8, we get $\varphi(A) \in \mathcal{M}'$. Hence $\varphi(A) \in (\mathcal{M} + R) \cup (\mathcal{N} + R)$.

Now, we assume that $\varphi(\ell_k)$ [resp. ℓ'_d] is contained in a line in $AG(\mathcal{M}')$ for $k = 1, 2$ [resp. all $d \in \mathbb{D}^*$]. Let $X, Y \in \ell_1 \cup \ell_2 \cup_{d \in \mathbb{D}^*} \ell'_d$ with $X \neq Y$. Then $X \sim Y$ and hence $\varphi(X) \sim \varphi(Y)$ and $\varphi(X) \neq \varphi(Y)$. By (79), we have $\varphi(A) \sim \varphi(X)$ and $\varphi(A) \sim \varphi(Y)$. Clearly, $\mathcal{M} + R$ or $\mathcal{N} + R$ contains $\lceil (|\mathbb{D}| + 1)/2 \rceil$ elements of the set $\{\varphi(\ell_1), \varphi(\ell_2), \varphi(\ell'_d), d \in \mathbb{D}^*\}$. Without loss of generality, we assume that $\mathcal{M} + R$ contains $\lceil (|\mathbb{D}| + 1)/2 \rceil$ elements of the set $\{\varphi(\ell_1), \varphi(\ell_2), \varphi(\ell'_d), d \in \mathbb{D}^*\}$. Then, $\mathcal{M} + R$ contains $(|\mathbb{D}| - 1)\lceil (|\mathbb{D}| + 1)/2 \rceil + 1$ distinct points in $\varphi(\ell_1) \cup \varphi(\ell_2) \cup_{d \in \mathbb{D}^*} \varphi(\ell'_d)$. Since $|\mathbb{D}'| \leq (|\mathbb{D}| - 1)\lceil (|\mathbb{D}| + 1)/2 \rceil$, every line in $AG(\mathcal{M} + R)$ contains at most $(|\mathbb{D}| - 1)\lceil (|\mathbb{D}| + 1)/2 \rceil$ points. Thus $AG(\mathcal{M} + R)$ contains at least three noncollinear points. It follows from (79) that $\varphi(A)$ are adjacent with three noncollinear points in $AG(\mathcal{M} + R)$. Applying Lemma 2.8, we obtain $\varphi(A) \in \mathcal{M} + R$.

Therefore, we always have $\varphi(A) \in (\mathcal{M} + R) \cup (\mathcal{N} + R)$ for any $A \in \begin{pmatrix} \mathbb{D}^{k \times n} \\ 0 \end{pmatrix}$. Then

$$\varphi \left(\begin{pmatrix} \mathbb{D}^{k \times n} \\ 0 \end{pmatrix} \right) \subseteq (\mathcal{M} + R) \cup (\mathcal{N} + R).$$

Taking $k = m$, we get (77). □

Remark 5.10 Let $n, m, p, q \geq 2$ be integers. By [23, Theorems 1.2 and 1.3], it is easy to see that every degenerate graph homomorphism $\varphi : \mathbb{F}_q^{m \times n} \rightarrow \mathbb{F}_q^{p \times q}$ is a (vertex) colouring. When $\mathbb{D} = \mathbb{D}' = \mathbb{F}_2$ or \mathbb{F}_3 , Both Theorem 4.1 and Corollary 5.7 still hold (cf. [23, 13, 17]). However, when $q \leq 3$ with $q < q'$, it is an open problem to characterize the non-degenerate graph homomorphisms from $\mathbb{F}_q^{m \times n}$ to $\mathbb{F}_{q'}^{p \times q}$.

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